Analogs in computability of combinatorial cardinal characteristics

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Set theory setting

- Several cardinal characteristics are based on cardinals of subclasses of $[\omega]^\omega$ (the infinite subsets of ω) viewed up to almost equality.
- One of them is called the almost disjointness number, denoted $\mathfrak{a}$.
- This is the minimal size of a maximal almost disjoint (MAD) family of subsets of ω.

Let $\mathfrak{b}$ be the unbounding number (the least size of a class of functions on ω that is not dominated by a single function).

Fact

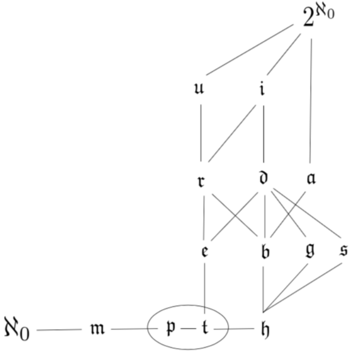
$\mathfrak{b} \leq \mathfrak{a}$.

See e.g. Logic Blog '19 for a proof of this well known fact.

Further cardinal characteristics based on properties of subsets of $[\omega]^\omega$ modulo almost equality:

- the ultrafilter number $\mathfrak{u}$ is the least size of a set with upward closure a free ultrafilter on ω,
- the tower number $\mathfrak{t}$ is the minimum size of a linearly ordered subset of $[\omega]^\omega$ that can't be extended by putting a new element below all given elements,
- the independence number $\mathfrak{i}$ (least size of a maximal independent set).

Diagram of ZFC inequalities (Soukup, 2018)



$\mathfrak{r}$ and $\mathfrak{s}$ are the unreaping and splitting numbers, respectively. Their analogs in computability have been studied (BBNN, 14). $\mathfrak{e}$ is the escaping number due to Brendle and Shelah. Its analog has been considered in computability theory by Valverde and Tveite (2017).

Analogous mass problems in computability

For a set F, $F^{[n]}$ denotes the column $\{x \colon \langle x, n\rangle \in F\}$.
We will usually denote this by F_n.

- The basic objects are collections of infinite recursive sets in the context of almost inclusion.
- Such a collection $\mathbf{C}$ is encoded by a set F such that
$$\mathbf{C} = \mathbf{C_F} = \{F^{[n]} \colon n \in \mathbb{N}\}.$$
- Recall that a mass problem is a set of functions $\mathbb{N} \to \mathbb{N}$.
- We view properties of such collections as mass problems, consisting of the characteristic functions of the encoding sets F.
- We compare their complexity via Muchnik reducibility $\leq_w$ and the stronger, uniform Medvedev reducibility $\leq_s$.

The mass problem $\mathcal{A}$ of MAD sets

We will often identify a set $F \subseteq \mathbb{N}$ and the collection $\mathbf{C}_F = \{F_n : n \in \mathbb{N}\}$ of recursive sets described by F.

We say that $F \subseteq \mathbb{N}$ is a almost disjoint, AD in brief, if

each F_n is infinite, and $F_n \cap F_k =^* \emptyset$ for $n \neq k$.

Definition (Analog of almost disjointness number)

- The mass problem $\mathcal{A}$ is the class of sets F such that $\mathbf{C}_F$ is maximal almost disjoint (MAD) for the recursive sets.
- Namely, $\mathbf{C}_F$ is AD, and for each infinite recursive set R there is n such that $R \cap F_n$ is infinite.

No MAD set F is computable

Proposition

No MAD set F is computable.

Suppose F is AD and computable.

Let $r_{-1} = 0$, and r_n be the least number $r > r_{n-1}$ such that $r \in F_n - \bigcup_{i<n} F_i$.

Then the computable set $R = \{r_0, r_1, \ldots\}$ shows that F is not MAD.

The mass problem $\mathcal{T}$ of maximal towers

We say that $G \subseteq \mathbb{N}$ is a tower (or $\mathbf{C}_G$ is a tower) if for each n we have $G_{n+1} \subseteq^* G_n$ and $G_n - G_{n+1}$ is infinite.

Definition (Analog of tower number $\mathfrak{t}$)

- The mass problem $\mathcal{T}$ is the class of sets G such that $\mathbf{C}_G$ is a tower that is maximal in the recursive sets.
- Namely, for each infinite recursive set R there is n such that $R - G_n$ is infinite.

$\mathcal{A} \equiv_s \mathcal{T}$ (Medvedev equivalent)

To check that $\mathcal{A} \leq_s \mathcal{T}$, given a set G let $\mathrm{Diff}(G)$ be the set D such that $D_n = G_n - G_{n+1}$. Clearly the operator Diff can be seen as a Turing functional.
If G is a maximal tower then $D = \mathrm{Diff}(G)$ is MAD. For, if R is infinite recursive then $R - G_n$ is infinite for some n, and hence $R \cap D_i$ is infinite for some $i < n$.

For $\mathcal{T} \leq_s \mathcal{A}$, given a set F let $G = \mathrm{Cp}(F)$ be the set such that

$$x \in G_n \leftrightarrow \forall i < n\,[x \notin F_n].$$

Again Cp is a Turing functional.
If F is AD then G is a tower, and if F is MAD then G is a maximal tower.

Each non-low oracle computes a set in $\mathcal{T}$

Theorem

$\mathcal{T}$ (the mass problem of max. towers) $\leq_s$ NonLow.

Proof. Let x, y, z denote binary strings; we identify x with the number $1x$ via the binary expansion. Define a Turing functional Φ for the Medvedev reduction: $\Phi^Z = G$, where for each n

$$G_n = \{x \colon\ n \leq s := |x| \wedge Z'_s \upharpoonright n = x \upharpoonright n\}.$$

It is clear that for each n we have $G_{n+1} \subseteq^* G_n$ and $G_n - G_{n+1}$ is infinite. Also, for each n, for large s the string $Z'_s \upharpoonright n$ settles, so G_n is computable.

Suppose now that R is an infinite set such that $R \subseteq^* G_n$ for each n. Then $Z'(k) = \lim_{x \in G, |x| > k} x(k)$, and hence $Z' \leq_T R'$. So if $Z \in$ NonLow then R cannot be computable, and hence $\Phi^Z \in \mathcal{T}$.

C.e. MAD set by a finitary priority construction

Theorem

For each incomputable c.e. set A, there is a c.e. MAD set $F \leq_T A$. This shows that a c.e., totally low set is computable.

Let $V_{2e} = W_e$ and $V_{2e+1} = \mathbb{N}$ for each e. Build auxiliary c.e. set $S \leq_{\mathrm{T}} A$. Then let $F \leq_T A$ be defined by $F^{[e]} = S^{[2e]} \cup S^{[2e+1]}$.

$$P_n : V_e - \bigcup_{i<n} S_i \text{ infinite } \Rightarrow |S_e \cap V_e| \geq k.$$

At stage s we say that P_n is satisfied if $|S_{e,s} \cap V_{e,s}| \geq k$.

Construction.

Stage $s > 0$. For each $n < s$ such that P_n is not satisfied, $n = \langle e, k \rangle$, if there is $x \in V_{e,s} - \bigcup_{i<n} S_{i,s}$ such that $x > \max(S_{e,s-1})$, $x \geq 2n$ and $A_s \upharpoonright x \neq A_{s-1} \upharpoonright x$, then put $\langle x, e \rangle$ into S (i.e., put x into S_e).

Indices for columns of a MAD hard to compute

A characteristic index for a set M is an e such that $\chi_M = \varphi_e$.

Proposition

Suppose F is MAD. Then $\emptyset'$ is not able to compute, from input n, a characteristic index for F_n.

Proof.

Assume so. Then there is a computable function f such that $\varphi_{\lim_s f(n,s)}$ is the characteristic function of F_n.

Let $\widehat{F}$ be defined as follows. Given n, x , compute the least $s > x$ such that $\varphi_{f(n,s),s}(x)\downarrow$. If the value is not 0 put x into $\widehat{F}_n$.

Clearly $\widehat{F}$ is computable. Since $F_n =^* \widehat{F}_n$ for each n, the set $\widehat{F}$ is MAD, contradicting the fact obtained above. $\square$

Totally low oracles

Definition

We call an oracle L totally low if whenever Φ_e^L is computable then $\emptyset'$ can compute from e an index for Φ_e^L.

In other words, there is a functional Γ a such that

$$\Phi_e^L \text{ is computable} \Rightarrow \Gamma(\emptyset'; e) \text{ is an index for it, i.e., } \Phi_e^L = \varphi_{\Gamma(\emptyset'; e)}.$$

- Exercise: totally low implies low.
- The proposition above implies that no totally low set computes a MAD set.

Not computing a MAD

Proposition

Suppose L is Δ^0_2 and 1-generic. Then L is totally low, and hence computes no MAD.

So we have:

$$\text{1-generic } \Delta^0_2 \Rightarrow \text{totally low} \Rightarrow \text{computes no MAD} \Rightarrow \text{low.}$$

At the present stage of this work in progress, we only know that the last arrow does not reverse. (Use a low, noncomputable c.e. set.)

The mass problem $\mathcal{U}$

Definition (Analog of the ultrafilter number $\mathfrak{u}$)

The mass problem $\mathcal{U}$ consists of the sets F such that

- $F_n \supset^* F_{n+1}$ and $F_n - F_{n+1}$ infinite
- for each recursive set R there is n such that $F_n \subseteq^* R$ or $F_n \subseteq^* \overline{R}$.

We say that F (or, more precisely $\mathcal{C}_F$) is an ultrafilter base (UFB) within the recursive sets.

Fact

$\mathcal{T} \supseteq \mathcal{U}$, that is, each UFB is a maximal tower.
So we trivially have $\mathcal{T} \leq_s \mathcal{U}$ via the identity reduction.

Example: take any r-maximal set C. By definition of r-maximality, the recursive sets R such that $R \cup C$ is cofinite form an ultrafilter. We can obtain an ultrafilter base $\mathbf{C}_F$ for some $F <_T \emptyset''$.

Proposition (warmup)

No ultrafilter base F is computably dominated.

Proof.

Let $f(n)$ be the least number $> n$ in $\bigcap_{i<n} F_i$. Then $f \leq_T F$.
Assume that there is a computable function $p \geq f$. The conditions $n_0 = 1$ and $n_{k+1} = p(n_k)$ define a computable sequence.
So the set

$$E = \bigcup_i [n_{2i}, n_{2i+1})$$

is computable.
Clearly $F_n \not\subseteq^* E$ and $F_n \not\subseteq^* \overline{E}$ for each n. So F is not an ultrafilter base. □

Highness and mass problems

Our aim is to show that ultrafilter bases have exactly the high degrees. How do we formulate a version of this for strong reductions?

- Let DomFcn denote the mass problem of functions h that dominate every computable function, and also satisfy $h(s) \geq s$ for all s.
- Let $\mathrm{Tot} = \{e \colon \phi_e \text{ is total}\}$. Note that F is high iff $\mathrm{Tot} \leq_T F'$.
- The approximations to Tot are the $\{0, 1\}$-valued binary functions f such that $\lim_s f(e, s) = \mathrm{Tot}(e)$.

Fact (Martin, morally)

DomFcn is strongly equivalent to

the mass problem of approximations to Tot.

Classifying the complexity of ultrafilter bases

Theorem

The mass problem DomFcn of dominating functions

is Medvedev equivalent to

the mass problem $\mathcal{U}$ of ultrafilter bases.
In particular, the degrees of UFB are exactly the high degrees.

Proof of DomFcn $\leq_s \mathcal{U}$ is inspired by a proof of Jockusch (1972) that any family of sets containing exactly the computable sets must have high degree.

Proof of DomFcn $\geq_s \mathcal{U}$

Let $\langle \psi_e \rangle_{e \in \mathbb{N}}$ be an effective listing of the $\{0, 1\}$ valued partial computable functions defined on an initial segment of $\mathbb{N}$. Let $V_{e,k} = \{x \colon \psi_e(x) = k\}$.

Let $T = \{0, 1, 2\}^{<\infty}$. Uniformly in $\alpha \in T$ we will define a possibly finite c.e. set S_α enumerated in an increasing fashion.

Let $S_{\emptyset,s} = [0, s)$. If we have defined (at stage s) the set $S_\alpha = \{r_0 < \ldots < r_k\}$, let $\widetilde{S}_\alpha$ contain the numbers of the form r_{2i}.

- Let $S_{\alpha 2} = \widetilde{S}_\alpha$.
- Let $S_{\alpha k} = \widetilde{S}_\alpha \cap V_{e,k}$ for $k = 0, 1$, $e = |\alpha|$.

Define a uniform list of Turing functionals Γ_e so that the sequence $\langle \Gamma_e^h(t) \rangle_{t \in \mathbb{N}}$ is nondecreasing, for each e and each oracle function h such that $h(s) \geq s$ for each s. We will let $F_e = \{\Gamma_e^h(t) \colon t \in \mathbb{N}\}$.

Definition of Γ_e. Given an oracle function h, we will write a_s for $\Gamma_e^h(s)$. Let $a_0 = 0$. Suppose $s > 0$ and a_{s-1} has been defined.

Let $\alpha \in T$ be the leftmost string of length e such that there is an $x \in S_{\alpha,h(s)}$ with $x > a_{s-1}$. Choose x least for α and let $a_s = x$. If there is no such α let $a_s = a_{s-1}$.

Verification. Suppose h is a dominating function. Then for each e we have $F_e =^* S_\alpha$, where α is leftmost string of length e such that S_α is infinite.

Theorem

There is a co-c.e. ultrafilter base.

References:

- Logic Blog 2019, Section 7
- These slides available on request
- Rupprecht thesis and paper (2010)
- Brendlle, Brooke, Ng, Nies 2014: surveyish paper on Cichon diagram in computability
- Greenberg, Kuyper, Turetsky ta: Weihrauch reductions