CS3230 Semester 2 2024/2025

Design and Analysis of Algorithms

Tutorial 01
Introduction and Asymptotic Analysis
For Week 02

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1 Notes

CS3230 tutorial format is as follows: We will consider a few questions per tutorial. For **each question**, we may ask a student to solve it. A **reasonable** attempt for that question will earn the student participation points. TA will give each student at least two chances over the semester. As there are $11-12$ tutorials, and around 5 questions per tutorial, each student should be able to get enough chances if they are coming regularly to the tutorials.

2 Lecture Review: Asymptotic Analysis

We say[1] that $f \in O(g)$ or $f = O(g)$ or $f(n) \in O(g(n))$ or $f(n) = O(g(n))$ if $\exists c, n_0 > 0$ such that $\forall n \geq n_0, 0 \leq f(n) \leq c \cdot g(n)$.

Informally, in words, the above says that (function) g is an upper bound on (function) f. This is the Big O worst-case time complexity analysis that you have learned since earlier courses, for example, from CS2040/C/S.

The four other asymptotic notations $\Omega, \Theta, o, \omega$ along with O, can be summarised as in the following table.

[1]We are fine with either notation although we prefer $f(n) \in O(g(n))$ notation.

We say	if $\exists c, c_1, c_2, n_0 > 0$ such that $\forall n \geq n_0$	In other words
$f(n) \in O(g(n))$	$0 \leq f(n) \leq c \cdot g(n)$	g is an **upper** bound on f
$f(n) \in \Omega(g(n))$	$0 \leq c \cdot g(n) \leq f(n)$	g is a **lower** bound on f
$f(n) \in \Theta(g(n))$	$0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$	g is a **tight** bound on f

We say	if $\forall c > 0$, $\exists n_0 > 0$ such that $\forall n \geq n_0$	In other words
$f(n) \in o(g(n))$	$0 \leq f(n) < c \cdot g(n)$	g is a **strict upper** bound on f
$f(n) \in \omega(g(n))$	$0 \leq c \cdot g(n) < f(n)$	g is a **strict lower** bound on f

3 Tutorial 01 Questions

Q1). Assume $f(n), g(n) > 0$, show:

(a) $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \Rightarrow f(n) \in o(g(n))$ — this has already been shown in lec01b.

(b) $\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \Rightarrow f(n) \in O(g(n))$

By definition of limit, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = z$, means
$\forall \epsilon > 0, \exists n_0 > 0$, such that $\forall n \geq n_0$,
$\frac{f(n)}{g(n)} \leq z + \epsilon$.

Hence, for $c = z + \epsilon, \exists n_0 > 0$, such that $\forall n \geq n_0$,
$f(n) \leq (z + \epsilon) \cdot g(n) = c \cdot g(n)$.
$f(n) \in O(g(n))$.

(c) $0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \Rightarrow f(n) \in \Theta(g(n))$

The above and the remaining parts can be done similarly by looking at the value of the limits.

However, the simplest way to see this is to combine the explanation above for O and explanation below for Ω to conclude that $f(n) \in \Theta(g(n))$.

(d) $\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0 \Rightarrow f(n) \in \Omega(g(n))$

The simplest way to see this is to flip $f(n)$ vs $g(n)$, and use the explanation above (the limit will remain z as before), then use the complementarity property $g(n) \in O(f(n))$ iff $f(n) \in \Omega(g(n))$.

(e) $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \Rightarrow f(n) \in \omega(g(n))$

The simplest explanation is to flip $f(n)$ - the faster growing function vs $g(n)$ - the slower growing function, the limit $\lim_{n\to\infty} \frac{g(n)}{f(n)}$ will be 0, then use the complementarity property $g(n) \in o(f(n))$ iff $f(n) \in \omega(g(n))$.

Q2). Assume $f(n), g(n) > 0$, show:

(a) Reflexivity

- $f(n) \in O(f(n))$

  Taking $c = 1$ (any constant ≥ 1), $n_0 = 1$, we have $\forall n \geq n_0$,
  $f(n) \leq (1 \cdot f(n) = c \cdot f(n))$.

- $f(n) \in \Omega(f(n))$

  Taking $c = 1$ (any positive constant ≤ 1), $n_0 = 1$, we have $\forall n \geq n_0$,
  $(c \cdot f(n) = 1 \cdot f(n)) \leq f(n)$.

- $f(n) \in \Theta(f(n))$

  Taking $c_1 = 1$, $c_2 = 1, n_0 = 1$, we have $\forall n \geq n_0$,
  $(c_1 \cdot f(n) = 1 \cdot f(n)) \leq f(n) \leq (1 \cdot f(n) = c_2 \cdot f(n))$.

(b) Transitivity

- $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$ implies $f(n) \in O(h(n))$

  $f(n) \in O(g(n))$ means there exist $c_{fg} > 0, n_{0fg} > 0$ such that $\forall n \geq n_{0fg}$, $f(n) \leq c_{fg} \cdot g(n)$.
  $g(n) \in O(h(n))$ means there exist $c_{gh} > 0, n_{0gh} > 0$ such that $\forall n \geq n_{0gh}$, $g(n) \leq c_{gh} \cdot h(n)$.
  Taking $c = c_{fg} \cdot c_{gh}, n_0 = max(n_{0fg}, n_{0gh})$, we have $\forall n \geq n_0$,
  $f(n) \leq c \cdot h(n)$.
  Hence, $f(n) \in O(h(n))$.

- Do the same for Ω, Θ, o, ω

  Same as above, just change $\leq$ to $\geq, =, <, >$, respectively. Here, $=$ for Θ denotes that we need to do both $\geq$ and $\leq$ bounds.

(c) Symmetry

- $f(n) \in \Theta(g(n))$ iff $g(n) \in \Theta(f(n))$

  Suppose $f(n) \in \Theta(g(n))$.
  Thus, there exist $c_1, c_2, n_0 > 0$ such that $\forall n \geq n_0$,
  $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$.

  $f(n) \leq c_2 \cdot g(n)$.
  Hence, $\frac{1}{c_2} \cdot f(n) \leq g(n)$ (divide LHS and RHS by c_2).

  $c_1 \cdot g(n) \leq f(n)$.
  Hence, $g(n) \leq \frac{1}{c_1} \cdot f(n)$ (divide LHS and RHS by c_1).

  Taking $c_1' = \frac{1}{c_2}, c_2' = \frac{1}{c_1}$ and with the same n_0, we have $\forall n \geq n_0$,
  $c_1' \cdot f(n) \leq g(n) \leq c_2' \cdot f(n)$.
  Thus, $g(n) \in \Theta(f(n))$.

(d) Complementarity

- $f(n) \in O(g(n))$ iff $g(n) \in \Omega(f(n))$

  Suppose $f(n) \in O(g(n))$.
  Thus, there exist $c, n_0 > 0$ such that $\forall n \geq n_0$,
  $f(n) \leq c \cdot g(n)$.

  Hence, $\frac{1}{c} \cdot f(n) \leq g(n)$ (divide LHS and RHS by c).
  Taking $c' = \frac{1}{c}$ and with the same n_0, we have $\forall n \geq n_0$,
  $c' \cdot f(n) \leq g(n)$.
  Thus, $g(n) \in \Omega(f(n))$.

- $f(n) \in o(g(n))$ iff $g(n) \in \omega(f(n))$

  Same as above, just change $\leq$ to $<$.

Q3). Which of the following statement(s) is/are True?

(a) $3^{n+1} \in O(3^n)$

True.
Taking $c = 3, n_0 = 1$, we have $\forall n \geq n_0$,
$3^{n+1} \leq 3 \cdot 3^n = c \cdot 3^n$.
$3^{n+1} \in O(3^n)$.

(b) $4^n \in O(2^n)$

False.
For all $c \geq 1, n_0 = c$, we have $\forall n \geq n_0$,
$(4^n = (2^2)^n = (2^n)^2 = 2^n \cdot 2^n) \geq c \cdot 2^n$, i.e., we cannot upper bound 4^n with constant times 2^n.

(c) $2^{\lfloor \log n \rfloor} \in \Theta(n)$ (we assume log is in base 2)

True.
Taking $c_1 = \frac{1}{2}, c_2 = 1, n_0 = 1$ ($\log 0$ is undefined), we have $\forall n \geq n_0$,
$(c_1 \cdot n = \frac{1}{2} \cdot n) \leq 2^{\lfloor \log n \rfloor} \leq (1 \cdot n = c_2 \cdot n)$.
$2^{\lfloor \log n \rfloor} \in \Theta(n)$.

(d) For constants $i, a > 0$, we have $(n + a)^i \in O(n^i)$

True.
Taking $c = 2^i, n_0 = a$, we have $\forall n \geq n_0$,
$(n + a)^i \leq (n + n)^i = (2n)^i = 2^i \cdot n^i = c \cdot n^i$.
$(n + a)^i \in O(n^i)$.

Q4). Which of the following statement(s) is/are True?
$2^{\log_2 n} \in$

(a) $O(n)$

(b) $\Omega(n)$

(c) $\Theta(\sqrt{n})$

(d) $\omega(n)$

$2^{\log_2 n} = n \in O(n)$, and also $n \in \Omega(n)$.
During the tutorial, TA can follow-up with this version:

Q4-variant). How about $2^{\log_4 n} \in$?

We can rewrite the logarithm from one base to another base: $\log_4 n = \frac{\log_2 n}{\log_2 4} = \frac{\log_2 n}{2}$.

Thus, $2^{\log_4 n} = 2^{\frac{\log_2 n}{2}} = (2^{\log_2 n})^{\frac{1}{2}} = n^{\frac{1}{2}} = \sqrt{n}$.

(a) $O(n)$

True.
$2^{\log_4 n} = \sqrt{n} \in O(n)$, taking $c = 1, n_0 = 1$.

(b) $\Omega(n)$

False

(c) $\Theta(\sqrt{n})$

True
$2^{\log_4 n} = \sqrt{n} \in \Theta(\sqrt{n})$, taking $c_1 = 1$ (or smaller), $c_2 = 1$ (or larger), $n_0 = 1$.

(d) $\omega(n)$

False

Remarks: Note that $\log_b n = c \cdot \log_a n$, where the constant $c = \frac{\log a}{\log b}$. Thus, $\log_b n$ and $\log_a n$ have the same order of growth for different a and b. But, when they used as exponents, like in this question+variant, they make a difference (be careful). A common mistake is to say $2^{\log_4 n} \in \Theta(n)$ — pick option 1 (ok) and 2 (wrong), missing option 3.

Q5). Rank the following functions by their order of growth.
(But if any two (or more) functions have the same order of growth, group them together).

- $f_1(n) = \log n$
- $f_2(n) = n!$
- $f_3(n) = 2^n + n$

  $f_3(n) \in \Theta(2^n)$.

- $f_4(n) = n^{2.3} + 16n + f_1(n)$

  $f_4(n) \in \Theta(n^{2.3})$.

- $f_5(n) = \log(n^2)$

  $f_5(n) = \log(n^2) = 2\log n$, hence the same order of growth as $f_1(n)$.

- $f_6(n) = \ln(n^{2n})$

  $f_6(n) = 2n\ln(n) \in \Theta(n \ln n)$.

$f_2(n) = n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$.
expand simplified $f_3(n) = 2^n = 2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2$.
can show by induction that for $n \geq 4, n! \geq \frac{n}{4} \cdot 2^n$.

Therefore, with respect to order of growth, we have $(f_1(n) = f_5(n)) \leq f_6(n) \leq f_4(n) \leq f_3(n) \leq f_2(n)$.