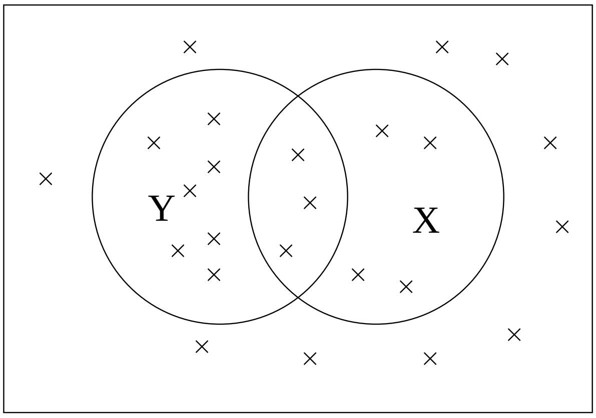
National University of Singapore
School of Computing
CS3243: Foundations of Artificial Intelligence
Solutions for Tutorial 9

Readings: AIMA Chapter 13 & 14

1. Based on the following Venn diagram, complete the joint probability distribution in the table on its right.



	X	$\neg X$
Y	$\frac{3}{24}$	$\frac{7}{24}$
$\neg Y$	$\frac{4}{24}$	$\frac{10}{24}$

Based on the joint probability distribution, find the following: $P(X)$, $P(Y)$, $P(\neg X)$, $P(\neg Y)$, $P(X|Y)$, $P(Y|X)$, $P(X|\neg Y)$, $P(\neg Y|X)$, $P(\neg X|Y)$, $P(Y|\neg X)$, $P(\neg X|\neg Y)$ and $P(\neg Y|\neg X)$.

$$
\begin{aligned}
P(X) &= \frac{3}{24} + \frac{4}{24} = \frac{7}{24} \\
P(Y) &= \frac{3}{24} + \frac{7}{24} = \frac{10}{24} \\
P(\neg X) &= \frac{7}{24} + \frac{10}{24} = \frac{17}{24} \\
P(\neg Y) &= \frac{4}{24} + \frac{10}{24} = \frac{14}{24} \\
P(X|Y) &= \frac{P(X \wedge Y)}{P(Y)} = \frac{\frac{3}{24}}{\frac{10}{24}} = \frac{3}{10} \\
P(Y|X) &= \frac{P(X \wedge Y)}{P(X)} = \frac{\frac{3}{24}}{\frac{7}{24}} = \frac{3}{7} \\
P(X|\neg Y) &= \frac{P(X \wedge \neg Y)}{P(\neg Y)} = \frac{\frac{4}{24}}{\frac{14}{24}} = \frac{2}{7} \\
P(\neg Y|X) &= \frac{P(X \wedge \neg Y)}{P(X)} = \frac{\frac{4}{24}}{\frac{7}{24}} = \frac{4}{7} \\
P(\neg X|Y) &= \frac{P(\neg X \wedge Y)}{P(Y)} = \frac{\frac{7}{24}}{\frac{10}{24}} = \frac{7}{10} \\
P(Y|\neg X) &= \frac{P(\neg X \wedge Y)}{P(\neg X)} = \frac{\frac{7}{24}}{\frac{17}{24}} = \frac{7}{17}
\end{aligned}
$$

$$\begin{aligned} P(\neg X|\neg Y) &= \frac{P(\neg X \wedge \neg Y)}{P(\neg Y)} = \frac{\frac{10}{24}}{\frac{14}{24}} = \frac{5}{7} \\ P(\neg Y|\neg X) &= \frac{P(\neg X \wedge \neg Y)}{P(\neg X)} = \frac{\frac{10}{24}}{\frac{17}{24}} = \frac{10}{17} \end{aligned}$$

Substituting the values of these conditional probabilities, verify the following:

$$\begin{aligned} P(X|Y) &= 1 - P(\neg X|Y) \\ P(X|\neg Y) &= 1 - P(\neg X|\neg Y) \\ P(\neg Y|X) &= \frac{P(X|\neg Y)P(\neg Y)}{P(X|\neg Y)P(\neg Y) + P(X|Y)P(Y)} \end{aligned}$$

We verify the above as follows:

$$\begin{aligned} 1 - P(\neg X|Y) &= 1 - \frac{7}{10} = \frac{3}{10} = P(X|Y) \\ 1 - P(\neg X|\neg Y) &= 1 - \frac{5}{7} = \frac{2}{7} = P(X|\neg Y) \\ \frac{P(X|\neg Y)P(\neg Y)}{P(X|\neg Y)P(\neg Y) + P(X|Y)P(Y)} &= \frac{\frac{2}{7} \cdot \frac{14}{24}}{\frac{2}{7} \cdot \frac{14}{24} + \frac{3}{10} \cdot \frac{10}{24}} = \frac{4}{7} = P(\neg Y|X) \end{aligned}$$

2. Consider a healthcare diagnosis system where symptoms are matched to illnesses. Discuss the limitations of first order logic that makes it unsuitable for the inference engine of this diagnosis system. Suggest a replacement method for implementing this inference engine.

   We cannot say that any of the following statements is true:

$$\forall x \; Flu(x) \Rightarrow Fever(x)$$
$$\forall x \; Fever(x) \Rightarrow Flu(x)$$

   even though there is a strong relationship between flu and fever. While it is technically possible to write statements like

$$\forall x \; Fever(x) \Rightarrow Flu(x) \vee Cold(x) \vee ...$$

   it is hardly possible or feasible to enumerate all possible illnesses. Further, it does not make it easy to "explain away" that the patient does not have flu and so he is likely to have a cold instead.

   A replacement method might be to use a Bayesian network.

3. Assume that 2% of the population in a country carry a particular virus. A test kit developed by a pharmaceutical firm is able to detect the presence of the virus from a patient's blood sample. The firm claims that the test kit has a high accuracy of detection in terms of the following conditional probabilities obtained from their quality control testing:

   P(the kit shows positive — the patient is a carrier) = 0.998
   P(the kit shows negative — the patient is not a carrier) = 0.996

If a patient is tested to be positive using this kit, what is the likelihood of a false positive (i.e., that he actually is not a carrier but the kit shows positive)?

Let X and $\neg X$ represent the test kit shows positive and negative, respectively. Let Y and $\neg Y$ represent the patient is a carrier and not a carrier, respectively. Then,

$$\begin{aligned} P(Y) &= 0.02 \\ \Rightarrow P(\neg Y) &= 1 - P(Y) = 1 - 0.02 = 0.98 \\ P(X|Y) &= 0.998 \\ P(\neg X|\neg Y) &= 0.996 \\ \Rightarrow P(X|\neg Y) &= 1 - P(\neg X|\neg Y) = 1 - 0.996 = 0.04 \end{aligned}$$

Applying Bayes' Rule,

$$\begin{aligned} P(\neg Y|X) &= \frac{P(X|\neg Y)P(\neg Y)}{P(X|\neg Y)P(\neg Y) + P(X|Y)P(Y)} \\ &= \frac{0.04 \times 0.98}{0.04 \times 0.98 + 0.99 \times 0.02} = 0.16. \end{aligned}$$

There is a 16% chance that when the test kit shows positive, the patient is not a carrier.

4. (Question 13.1 from AIMA) Show from first principles that

$$P(a|a \wedge b) = 1$$

.

$$\begin{aligned} P(a|a \wedge b) &= \frac{P(a \wedge a \wedge b)}{P(a \wedge b)} \\ &= \frac{P(a \wedge b)}{P(a \wedge b)} \\ &= 1 \end{aligned}$$

5. (Question 13.7 from AIMA) Show that the three forms of independence below:

   (a) $P(a|b) = P(a)$

   (b) $P(b|a) = P(b)$

   (c) $P(a \wedge b) = P(a)P(b)$

   are equivalent.

   To prove that (a), (b), and (c) are equivalent, we will prove (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), and (c)$\Rightarrow$ (a).

   (a) $\Rightarrow$ (b): Given $P(a|b) = P(a)$,

$$\begin{aligned} P(b|a) &= \frac{P(a|b)P(b)}{P(a)} \\ &= P(b) \end{aligned}$$

   (b) $\Rightarrow$ (c): Given $P(b|a) = P(b)$,

$$\begin{aligned} P(a \wedge b) &= P(b|a)P(a) \\ &= P(a)P(b) \end{aligned}$$

(c)⇒ (a): Given $P(a \wedge b) = P(a)P(b)$,

$$\begin{aligned} P(a|b) &= \frac{P(a \wedge b}{P(b)} \\ &= \frac{P(a)P(b)}{P(b)} \\ &= P(a) \end{aligned}$$

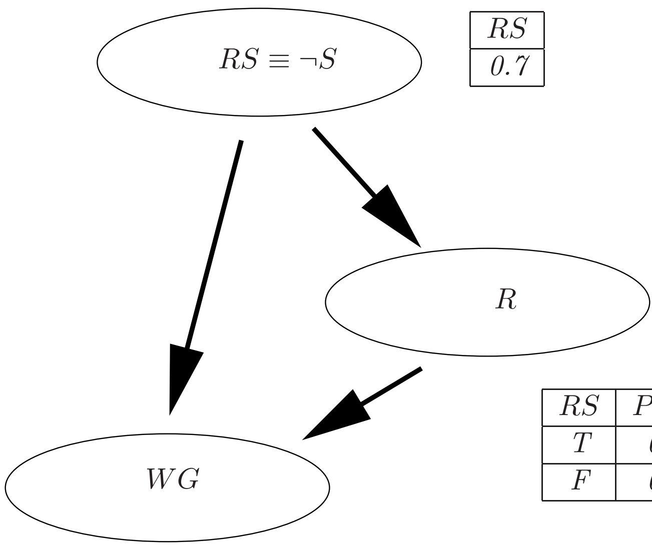
6. Assume that the following conditional probabilities are available:

P(Wet_Grass | Sprinkler $\wedge$ Rain)	0.95
P(Wet_Grass | Sprinkler $\wedge$ $\neg$ Rain)	0.9
P(Wet_Grass | $\neg$ Sprinkler $\wedge$ Rain)	0.8
P(Wet_Grass | $\neg$ Sprinkler $\wedge$ $\neg$ Rain)	0.1
P(Sprinkler | Rainy_Season)	0.0
P(Sprinkler | $\neg$ Rainy_Season)	1.0
P(Rain | Rainy_Season)	0.9
P(Rain | $\neg$ Rainy_Season)	0.1
P(Rainy_Season)	0.7

Construct a Bayesian network and determine the probability

$$P(\text{Wet\_Grass} \wedge \text{Rainy\_Season} \wedge \neg \text{ Rain} \wedge \neg \text{ Sprinkler}).$$

Let RS, S, R, and WG denote Rainy_Season, Sprinkler, Rain, and Wet_Grass respectively.



RS
0.7

RS	R	P(R)
T	T	0.8
T	F	0.1
F	T	0.95
F	F	0.9

RS	P(R)
T	0.9
F	0.1

$$\begin{aligned}
P(WG \wedge RS \wedge \neg R \wedge \neg S) &= P(WG|RS \wedge \neg R \wedge \neg S)P(RS \wedge \neg R \wedge \neg S) \\
&= P(WG|RS \wedge \neg R \wedge \neg S)P(\neg R|RS \wedge \neg S)P(RS \wedge \neg S) \\
&= P(WG|RS \wedge \neg R \wedge \neg S)P(\neg R|RS \wedge \neg S)P(\neg S|RS)P(RS) \\
&= 0.1 \times (1 - 0.9) \times (1 - 0.0) \times 0.7 \\
&= 0.007
\end{aligned}$$

7. An expert system called PROSPECTOR for use in geological exploration makes use of an inference mechanism similar to a Bayesian network. The following are two modified versions of its rule patterns:

If E1
Then H1 (P(H1 | E1), P(H1 | ¬E1))

If E2
and E3
Then H2 (P(H2 | E2∧ E3), P(H2 | E2∧ ¬E3), P(H2 | ¬E2∧ E3), P(H2 | ¬E2 ∧ ¬E3))

The following is a hypothetical set of PROSPECTORs rules (where we also use two letters to represent propositions for your easy working later)

If the igneous rocks in the region have a fine to medium grain size (Gr)
bThen they have a porphyritic texture (Tx) (0.6, 0.2)

If the igneous rocks in the region have a fine to medium grain size (Gr)
and they have a porphyritic texture (Tx)
Then the region is a hypabyssal environment (Hy) (0.88, 0.76, 0.52, 0.02)

If the region is a hypabyssal environment (Hy)
Then the region has a favourable level of erosion (Er) (0.75, 0.12)

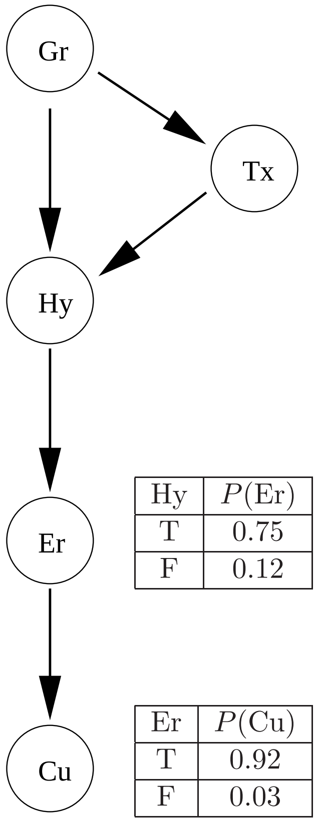
If the region has a favourable level of erosion (Er)
Then the region is favourable for copper deposits (Cu) (0.92, 0.03)

Construct a Bayesian network based on the above rules. Assume that a geologist could only ascertain with probability 0.15 that a regions igneous rocks have a fine to medium grain size. What is the probability that this region is favourable for copper deposits and has a favourable level of erosion, given that the region (1) has large grain size igneous rocks, (2) has non-porphyritic texture rocks, and (3) is a hypabyssal environment.

P(Gr)
0.15

Gr	P(Tx)
T	0.6
F	0.2

Gr	Tx	P(Hy)
T	T	0.88
T	F	0.76
F	T	0.52
F	F	0.02



Hy	P(Er)
T	0.75
F	0.12

Er	P(Cu)
T	0.92
F	0.03

Note that:

$$P(Cu|Er \wedge \neg Gr \wedge \neg Tx \wedge Hy) = P(Cu|Er)$$
$$P(Er|\neg Gr \wedge \neg Tx \wedge Hy) = P(Er| \wedge Hy)$$

Refer to the conditionalized version of the general product rule:

$$P(A, B|C) = P(A|B, C)P(B|C)$$

Thus,

$$\begin{aligned} P(Cu \wedge Er|\neg Gr \wedge \neg Tx \wedge Hy) &= P(Cu|Er \wedge \neg Gr \wedge \neg Tx \wedge Hy)P(Er|\neg Gr \wedge \neg Tx \wedge Hy) \\ &= P(Cu|Er)P(Er|Hy) \\ &= 0.92 \times 0.75 \\ &= 0.69 \end{aligned}$$