CS2109S Matrix calculus Cheatsheet

Notation

Notation	Meaning
$x, y, \epsilon \in \mathbb{R}$	scalar
$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\epsilon} \in \mathbb{R}^d$	vector
$\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{W} \in \mathbb{R}^{d\times n}$	matrix
$x_i(X_{ij})$	entries of vector(matrix)
c, k	#classes
d, m	#features
n	#samples
in, out	#input/output features
b, w_0	bias
l, ϵ	loss, error

Scalar calculus

The entries of vector and matrix follow these rules as well. $f, g : \mathbb{R} \to \mathbb{R}$ are functions of x. $a \in \mathbb{R}$.

Expression	Derivative w.r.t. x
a	0
$a \cdot f$	$a \cdot \frac{df}{dx}$
x^n	nx^{n-1}
$f + g$	$\frac{df}{dx} + \frac{dg}{dx}$
$f - g$	$\frac{df}{dx} - \frac{dg}{dx}$
$f \cdot g$	$f \cdot \frac{dg}{dx} + \frac{df}{dx} \cdot g$
$f(g(x))$	$\frac{df(u)}{du}\frac{du}{dx}$, let $u = g(x)$

Matrix calculus

Using denominator layout ($y \in \mathbb{R}, \boldsymbol{x}, \frac{\partial y}{\partial \boldsymbol{x}} \in \mathbb{R}^d$)

$$\frac{\partial y}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_d} \end{bmatrix}$$

Hessian formulation

For vector-valued functions $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}) : \mathbb{R}^d \to \mathbb{R}^c$

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial y_1}{\partial \boldsymbol{x}} & \frac{\partial y_2}{\partial \boldsymbol{x}} & \dots & \frac{\partial y_c}{\partial \boldsymbol{x}} \end{bmatrix}$$

Common vector derivatives

$\boldsymbol{b} \in \mathbb{R}^d$ and $\boldsymbol{A} \in \mathbb{R}^{d\times c}$ are not functions on $\boldsymbol{x} \in \mathbb{R}^d$

$\boldsymbol{f}(\boldsymbol{x})$	$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}$
$\boldsymbol{A}^T\boldsymbol{x}$	$\boldsymbol{A}$
$\boldsymbol{b}^T\boldsymbol{x}$	$\boldsymbol{b}$
$\boldsymbol{x}^T\boldsymbol{b}$	$\boldsymbol{b}$
$\boldsymbol{x}^T\boldsymbol{x}$	$2\boldsymbol{x}$
$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$	$2\boldsymbol{A}\boldsymbol{x}$

Chain rule in matrix form

Since matrix multiplication does not commute, the order of the derivatives matters in the chain rule.

For instance, given $y = \boldsymbol{b}^T(\boldsymbol{X}^T\boldsymbol{a})$. Let $\boldsymbol{h} = \boldsymbol{X}^T\boldsymbol{a}$ and $\boldsymbol{k} = \boldsymbol{X}\boldsymbol{b}$. Then,

$$\frac{\partial y}{\partial \boldsymbol{a}} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{a}}\frac{\partial y}{\partial \boldsymbol{h}} = \boldsymbol{X}\boldsymbol{b}, \quad \frac{\partial y}{\partial \boldsymbol{X}} = \frac{\partial y}{\partial \boldsymbol{k}}(\frac{\partial \boldsymbol{k}}{\partial \boldsymbol{X}})^T = \boldsymbol{a}\boldsymbol{b}^T.$$

The order the derivatives might vary, but it can be determined by a shape consistency check.

Matrix chain rule, for back propagation

Given $\boldsymbol{Y} = \boldsymbol{W}^T\boldsymbol{X}$, $\epsilon = l(\boldsymbol{Y})$ as a loss function.

- Derivative to update weight:

$$\frac{\partial \epsilon}{\partial \boldsymbol{W}} = \boldsymbol{X} \cdot (\frac{\partial \epsilon}{\partial \boldsymbol{Y}})^T$$

- Derivative to be carried to the previous layer:

$$\frac{\partial \epsilon}{\partial \boldsymbol{X}} = \boldsymbol{W} \cdot \frac{\partial \epsilon}{\partial \boldsymbol{Y}}$$

Note: Use the shape of matrices to determine the order.

Matrix derivatives

Scalar-by-Matrix: the shape of a scalar-by-matrix derivative is the same as that of the matrix.

For instance:

$$\frac{\partial y}{\partial \boldsymbol{X}} = \begin{bmatrix} \frac{\partial y}{\partial X_{11}} & \frac{\partial y}{\partial X_{12}} & \cdots & \frac{\partial y}{\partial X_{1n}} \\ \frac{\partial y}{\partial X_{21}} & \frac{\partial y}{\partial X_{22}} & \cdots & \frac{\partial y}{\partial X_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y}{\partial X_{m1}} & \frac{\partial y}{\partial X_{m2}} & \cdots & \frac{\partial y}{\partial X_{mn}} \end{bmatrix}$$

The shape of the resulting derivative is the same as the shape of $\boldsymbol{X}$.

Notation for models

Linear regression Hypothesis function: $h(\boldsymbol{w}^T\boldsymbol{x})$, where $\boldsymbol{x}, \boldsymbol{w} \in \mathbb{R}^d$.

Logistic regression Apply the same shape as above.

SVM Apply the same shape as above.

Neural Network One layer: $\boldsymbol{Y} = \boldsymbol{W}^T\boldsymbol{X}$, where $\boldsymbol{X} \in \mathbb{R}^{in\times n}, \boldsymbol{W} \in \mathbb{R}^{in\times out}, \boldsymbol{Y} \in \mathbb{R}^{out\times n}$.