NP-completeness

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Overview

Recap

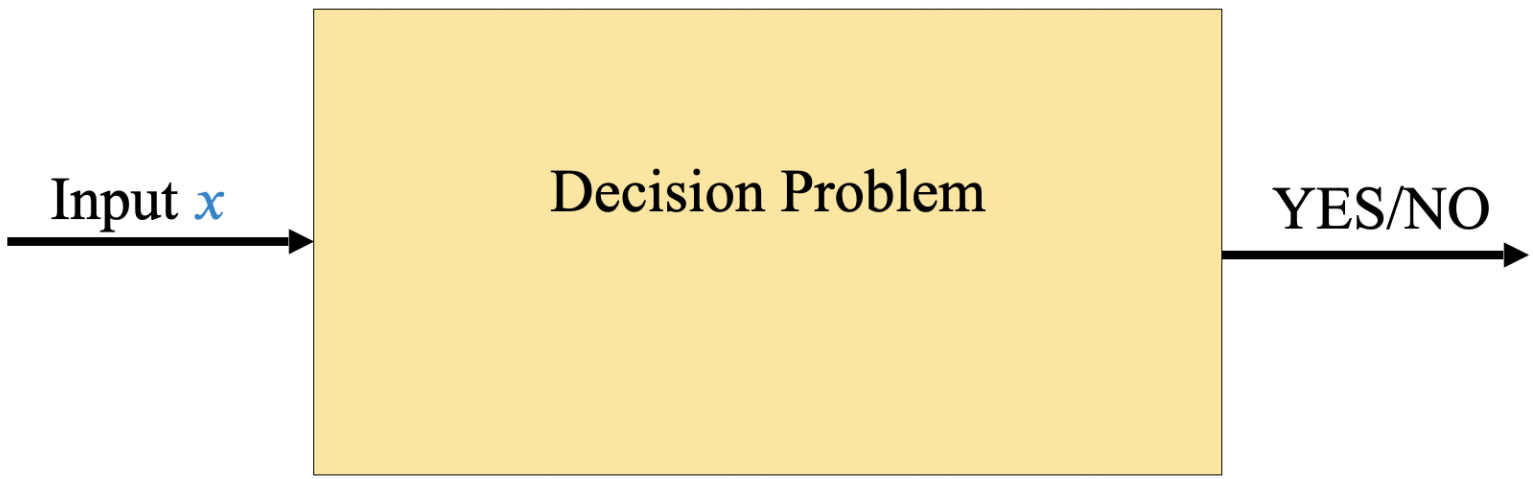
NP-completeness

- The History
- Searching versus Verifying
- NP class
- NP versus P
- NP-complete
- The first NP-complete problem
- Proving NP-completeness: $\texttt{C-SAT} \leq_p \texttt{CNF-SAT} \leq_p \texttt{3-SAT}$
- $\texttt{3-SAT} \leq_p \texttt{IS}$
- $\texttt{IS} \leq_p \texttt{VC}$
- $\texttt{VC} \leq_p \texttt{HS}$

Wrapping-Up

Recap: Decision Problems

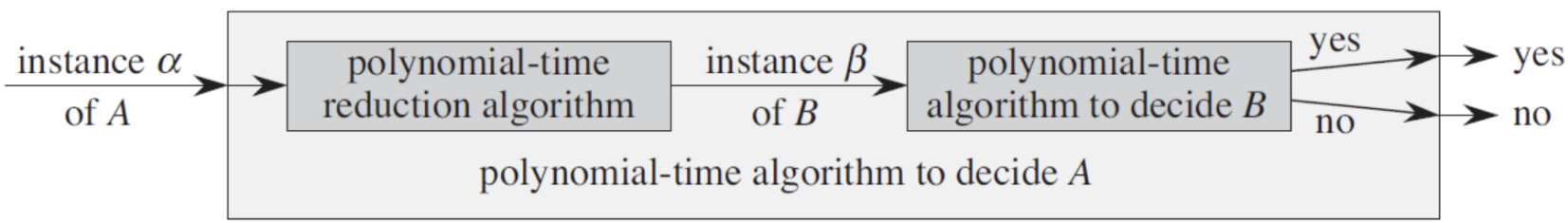
A **decision problem** is a function that maps an instance space I into the solution set { YES, NO }.



Recap: Reductions between Decision Problems

Given two decision problems A and B, a **polynomial-time reduction**[1] from A to B, denoted by $A \leq_p B$, is a transformation from instance α of A to instance β of B such that:

1. α is a YES-instance for A **iff** β is a YES-instance for B.
2. The transformation takes polynomial-time in the size of α.



[1] Also known as Karp reduction.

NP

NP — A class of problems

and how it came into existence

Going back to the 1960s

Efficient algorithm was found	No efficient algorithm till date
Shortest Path (SSSP)	Longest Path
Min Spanning Tree (MST)	Traveling Salesperson Problem
Euler tour	Hamiltonian cycle
Min Cut	Balanced Cut
VC on Tree	Vertex Cover (VC)
IS on Bipartite Graph	Independent Set (IS)
Bipartite Matching	3D Matching
Linear Programming (LP)	Integer Programming (IP)

It was quite surprising and even frustrating to be unable to find efficient algorithm for so many problems when their similar looking versions had very efficient algorithms.

Searching versus Verifying on 2 hard problems

`Longest-Path` **decision problem**

Given a graph G, does there exist a path of length $\geq k$?

Searching for a path of length $\geq k$ appears to be difficult.

Verifying if a given path is of length $\geq k$ is easy.

`Vertex-Cover` **decision problem**

Given a graph G, does there exist a vertex cover of size $\leq k$?

Searching for a subset of $\leq k$ vertices that is a vertex cover appears to be difficult.

Verifying if a given subset of $\leq k$ vertices is a vertex cover is easy.

Back to these hard problems

No efficient algorithm till date
Longest Path
Traveling Salesperson Problem
Hamiltonian cycle
Balanced Cut
Vertex Cover (VC)
Independent Set (IS)
3D Matching
Integer Programming (IP)

Searching for the solution is difficult
Verifying the solution (given the **short certificate**) is easy

NP class - Background

Given X: any decision problem and I: any (input) instance of X (which can be a YES-instance or a NO-instance),
a **verifier** for X is an algorithm A with output $\{$ YES, NO $\}$.

Thus, given input (I, s), where s is a certificate/proposed solution, this algorithm A can *verify* if the certificate s is right or wrong.

Formally: I is YES-instance of X **if and only if (iff)** there exists a **string** s such that A outputs YES on input (I, s).

To show that verification is easy, A must run in **polynomial-time**.

To show that s is short, $|s| \leq p(|I|)$ — p is a polynomial function.

NP class - Definition

NP: The set of all decision problems which have efficient (polynomial-time) **verifier**.

NP: "**Non-deterministic polynomial time**"

Example: `Hamiltonian-Cycle` (HC)

Given a graph G, does there exist a (simple) cycle that visits each vertex exactly once?

The certificate is the cycle itself.

The verifier checks whether it is a cycle (starts and ends at the same vertex) and visits each vertex once (perhaps by using a Direct Addressing Table (DAT) of `visited` flags). This runs in polynomial-time (in $O(|V|)$).

Hence, HC is in NP.

NP versus P

Recall:

NP: "**Non-deterministic polynomial time**"

NP: The set of all decision problems which have efficient (polynomial-time) **verifier**, i.e., 'easy to check'.

versus

P: "**PTIME**"

P: The set of all decision problems which have efficient (polynomial-time) **algorithm**, i.e., 'easy to solve'.

Is there any relation between P and NP?

NP class - for problems in P

Recall:

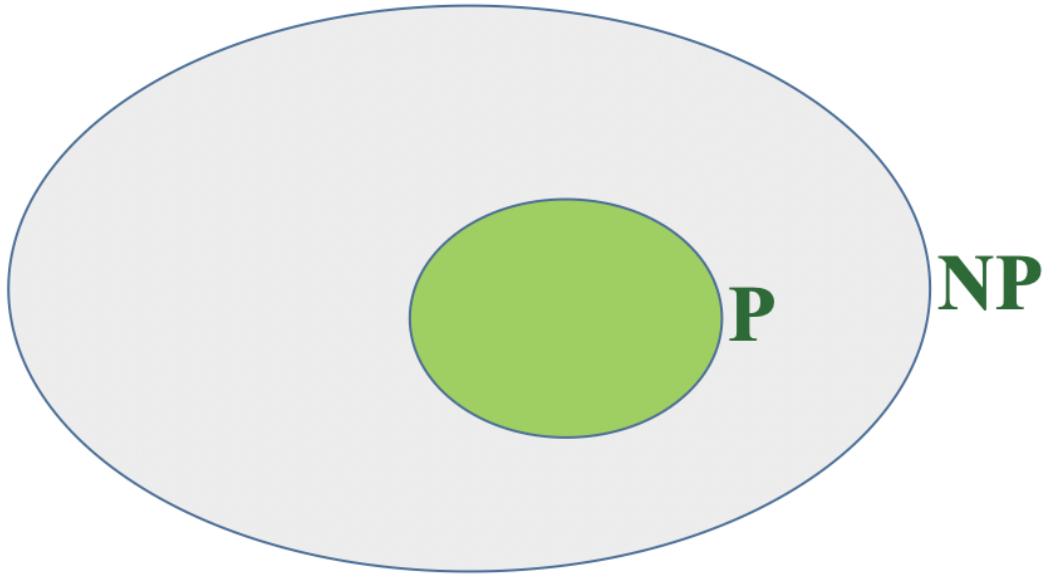
Given X: any decision problem and I: any (input) instance of X
(which can be a YES-instance or a NO-instance),
a **verifier** for X is an algorithm A with output $\{$ YES, NO $\}$.

If X is a problem in P,
let Q be a **polynomial-time** algorithm for **solving** X.

Thus, given input (I, s), where s is a certificate/proposed solution,
this algorithm A can ~~*verify* if the certificate s is right or wrong.~~
just ignore s, and execute the algorithm Q on input I.
If the answer is YES/NO, output YES/NO, respectively.

NP versus P - First Diagram

NP: Verifying a proposed solution versus P: Finding a solution



Open Question: Is **P = NP**?

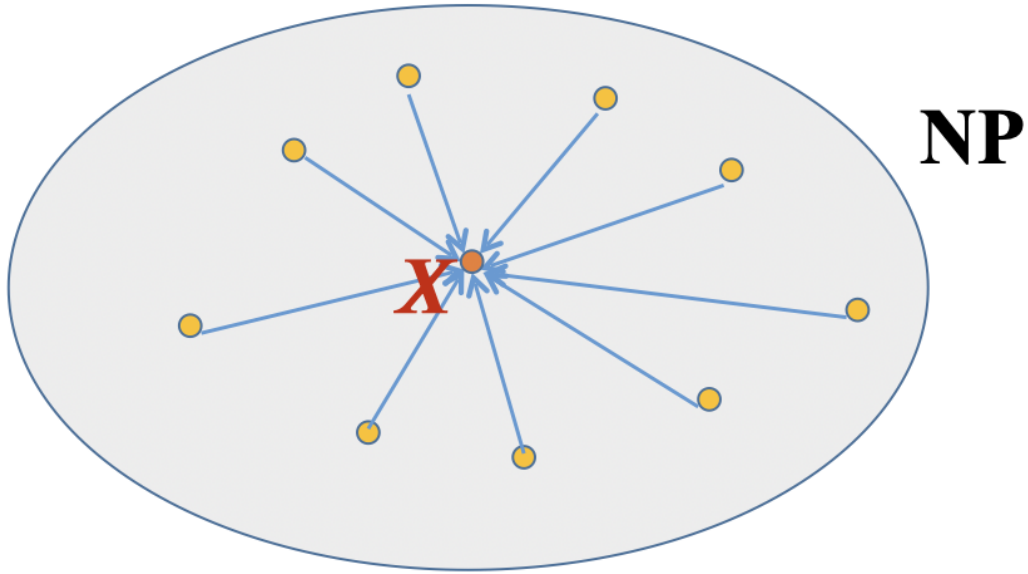
NP-complete

NP-complete — A(nother) class of problems

and how it came into existence

NP-complete - Definition

A problem X in NP class is in NP-complete if
for every $A \in$ **NP,** $A \leq_p X$.



PS: If X is not known (or not yet proven) to be in NP, then we say X is NP-hard.
(at least as hard as any other problem in NP).

Does any NP-complete problem exist?

It really needs **courage** to ask such a question and **great insight** to pursue its answer.

Because according to NP-complete definition: **Every problem**, known as well unknown problems, from the class NP has been reducible to this NP-complete problem.

Therefore, such an NP-complete problem would indeed be the hardest of all problems in NP.

Only such great questions in science lead to great inventions.

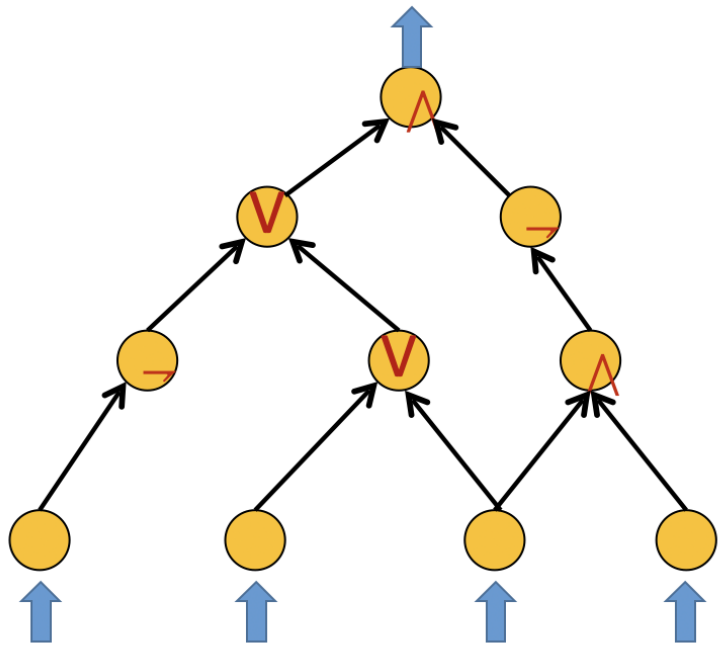
The first one: Circuit-Satisfiability (C-SAT)

Circuit-Satisfiability (C-SAT): [Cook and Levin, 1971]:

Given a DAG with vertices corresponding to AND, OR, NOT gates and n binary inputs, does there exist any binary input which gives output 1?

Why is C-SAT in NP?

The certificate is a binary input that gives output 1.



e.g., F,T,T,F for this example.

For every $A \in NP$, $A \leq_p$ C-SAT

Consider any problem $A \in$ NP.
What we know is that it has an efficient verifier, say Q.

Any verifier algorithm which outputs YES/NO can be represented as a C-SAT DAG where the internal vertices are the gates, the leaves are the binary inputs, and the output is 1/0 (or YES/NO).

So Cook & Levin essentially transform Q into the corresponding DAG and simulates Q on the instance I and proposed solution s.

PS: This is just a sketch.
Interested students should study Cook-Levin theorem in the future.

Satisfiability (CNF-SAT) - Definitions

Literal:
A Boolean variable or its negation, e.g., x_i, $\overline{x_i}$.

Clause:
A disjunction (OR) of literals, e.g., $C_j = x_i \vee \overline{x_2} \vee x_3$.

Conjunctive Normal Form (CNF):
a formula Φ that is a conjunction (AND) of clauses

CNF-SAT:
Given a CNF formula Φ, does it have a satisfying truth assignment?

3-Satisfiability (3-SAT) - Definitions

3-SAT is CNF-SAT where each clause contains exactly 3 (three) literals corresponding to different variables.

For example: $\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$

An unsatisfying assignment: $x_1 = \mathit{True}$ and $x_2 = x_3 = x_4 = \mathit{False}$.
PS: There are 4 other unsatisfying assignments for this Φ.

A satisfying assignment: $x_1 = x_2 = x_4 = \mathit{True}$ and $x_3 = \mathit{False}$.
PS: There are 10 other satisfying assignments for this Φ.

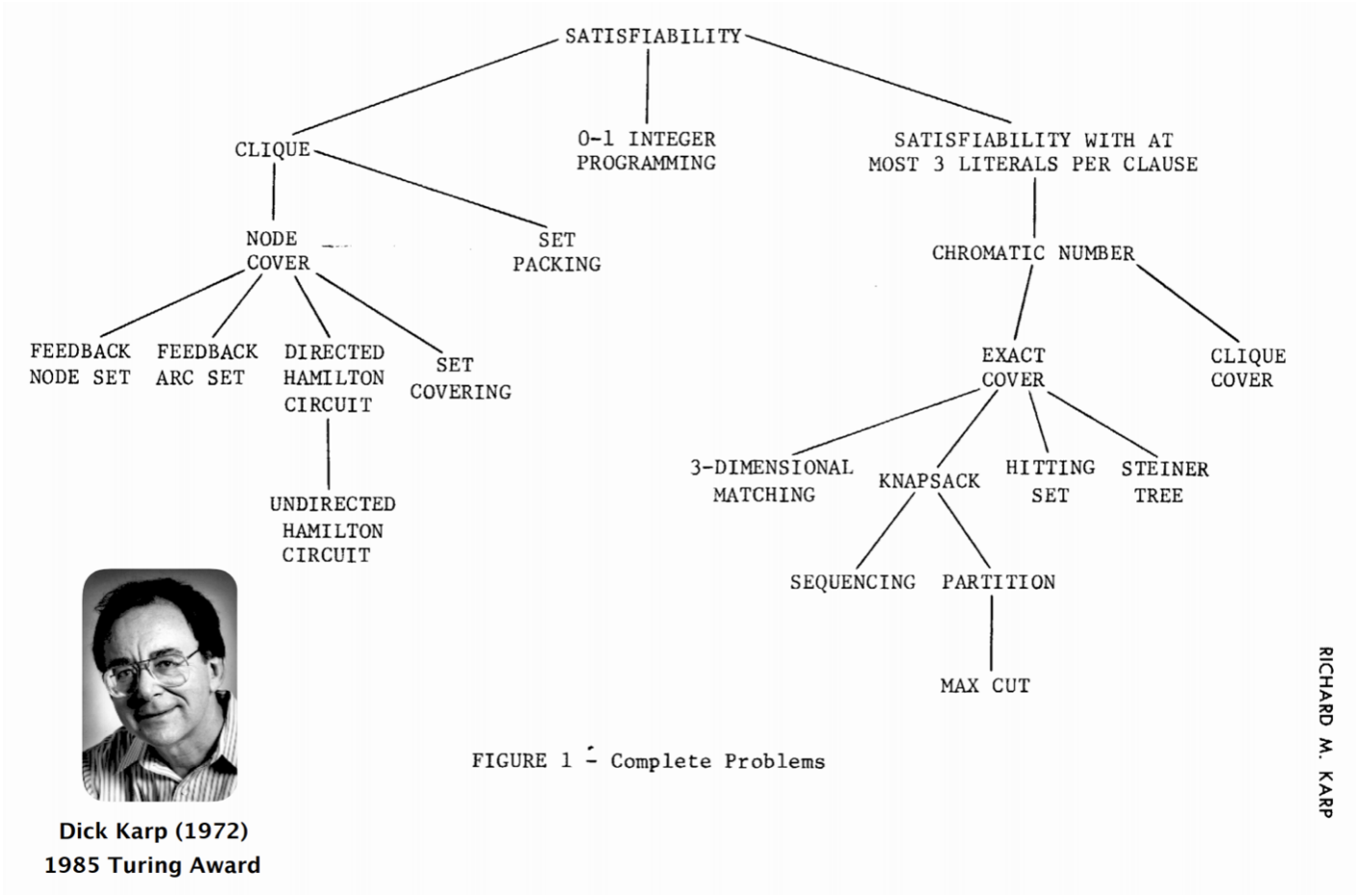
3-SAT is NP-complete

Earlier, we have been shown that C-SAT is NP-complete.

We are not proving these, but C-SAT $\leq_p$ CNF-SAT $\leq_p$ 3-SAT.

So, 3-SAT is NP-complete.

Reduction Proofs

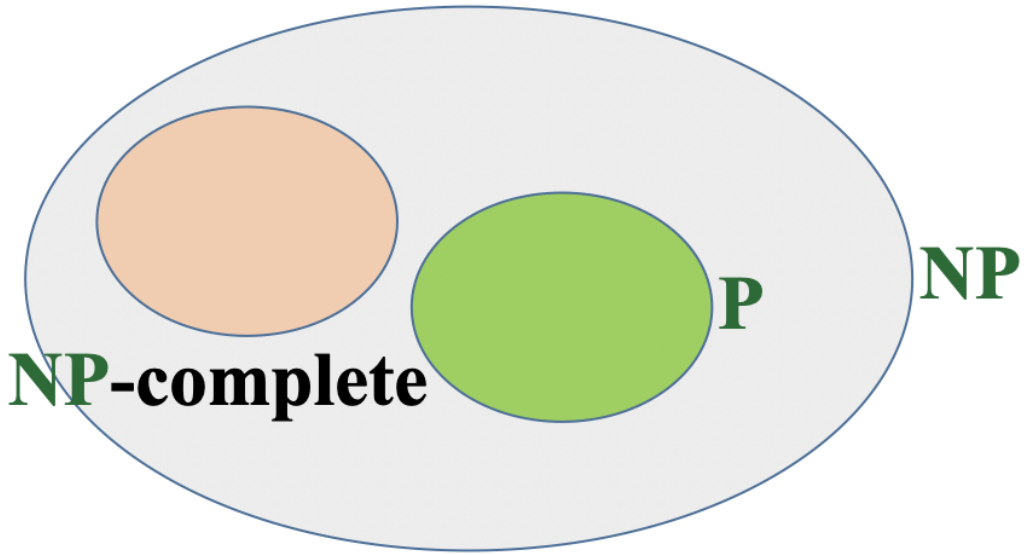


Dick Karp (1972)
1985 Turing Award

Our version (ongoing): `https://visualgo.net/en/reductions`

NP versus P - Second Diagram

Recall: A problem X in NP class is in NP-complete if **for every** $A \in$ **NP,** $A \leq_p X$.



Implication: If *any* NP-complete problem is solved in polynomial-time, then P = NP.

Question 1 at VisuAlgo Online Quiz

If you can prove P = NP (**or** P $\neq$ NP),
what will you get from Clay Mathematics Institute?

A). Turing award

B). USD $1 M

C). Nobel prize

D). Clay prize

E). Nothing

Question 2 at VisuAlgo Online Quiz

Which of the following decision problems is **not** NP-complete?

A). `C-SAT`

B). `CNF-SAT`

C). `3-SAT`

D). `2-SAT`

E). None of the above

Some Quotes

"I conjecture that there is no good algorithm for the traveling salesman problem. My reasons are the same as for any mathematical conjecture: (i) It is a legitimate mathematical possibility and (ii) I do not know." — Jack Edmonds (1966)

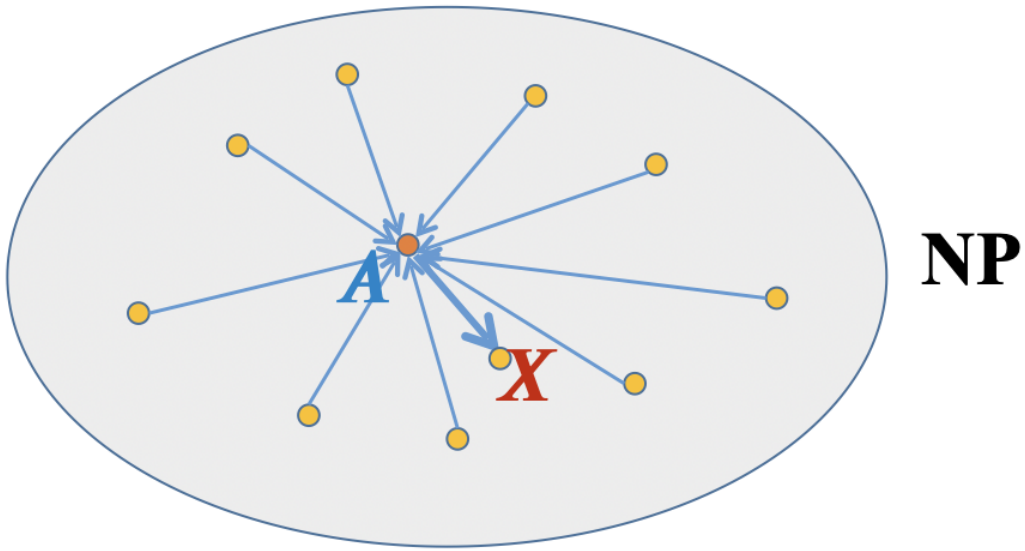
"If I had to bet now, I would bet that P $\neq$ NP. I estimate the half-life of this problem at 25–50 more years, but I would not bet on it being solved before 2100." — Robert Tarjan (2002)

"I think that in this respect I am on the loony fringe of the mathematical community: I think (not too strongly!) that P=NP and this will be proved within twenty years. Some years ago, Charles Read and I worked on it quite bit, and we even had a celebratory dinner in a good restaurant before we found an absolutely fatal mistake." — Béla Bollobás (2002)

How to show a problem to be NP-complete?

Let X be a problem which we wish to show to be NP-complete (e.g., in WA3 and/or final assessment). Follow these two steps:

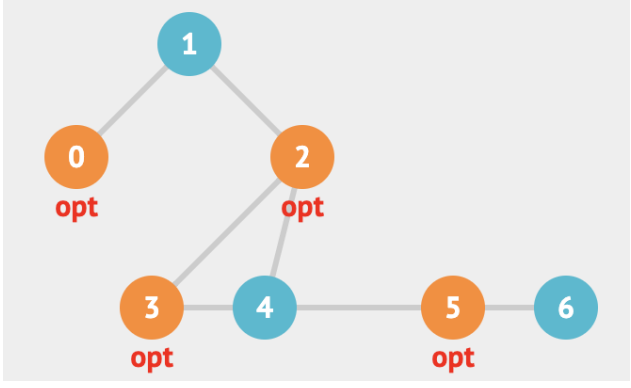
1. Show that $X \in$ NP (This is usually the easier one).
2. Pick a problem A which is already known to be NP-complete (having a library of problems that are NP-complete helps, i.e., `https://visualgo.net/en/reductions`);
   Then, show that $A \leq_p X$, i.e., X is NP-hard.



Independent-Set (IS)

Definition: Given an undirected graph $G = (V, E)$, a subset $X \subseteq V$ is said to be an **independent set** if for each $u, v \in X$, edge $(u, v) \notin E$.

Optimization version: compute `Max-Independent-Set` (`MIS`).
See `https://visualgo.net/en/mvc` (both are very related).



For example, MVC $= \{0, 2, 3, 5\}$ and MIS $= \{1, 4, 6\}$.

Decision version: Does there exist an independent set of size $\geq k$?

Prove IS is NP-complete

1. Show that IS $\in$ NP
   - The certificate is the independent set.
   - We can check if the size is $\geq k$ in $O(1)$, polynomial-time.
   - We can also check all edges $(u, v) \in E$, so that only at most u or v (but not both) is/are in the independent set (perhaps store the independent set information as a Hash Table or a DAT). This check can be done in $O(E)$, which is also polynomial-time.
2. Pick a problem A which is already known to be NP-complete; Then, show that $A \leq_p X$, i.e., X is NP-hard.
   - Any NP-complete problem that we know can be problem A.
   - But choose the one that leads to the easiest reduction.
   - We will pick 3-SAT.

3-SAT $\leq_p$ IS

Be careful of the direction of the reduction, it is from a **known** NP-complete problem, in this case 3-SAT to the (new) problem that you want to show to be NP-complete, in this case IS.

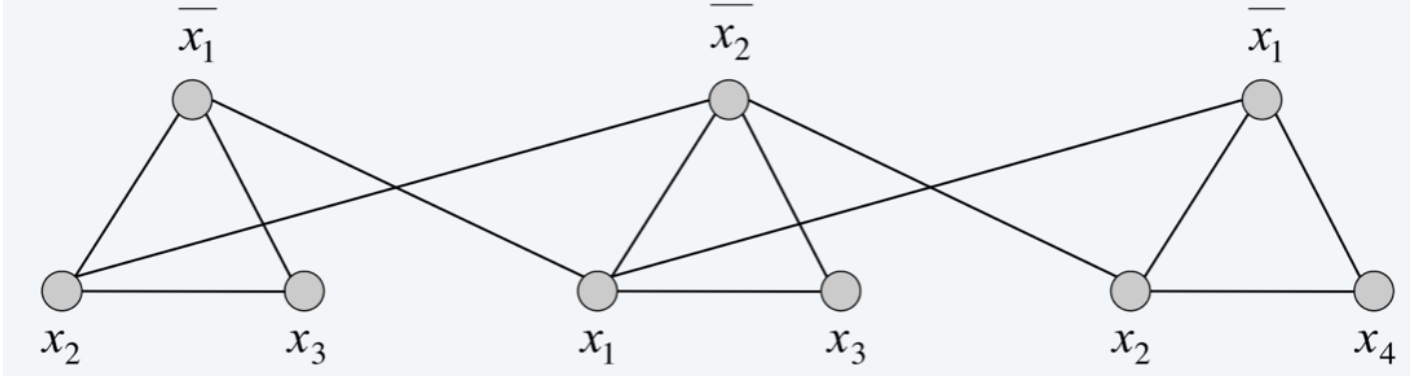
Given an instance Φ of 3-SAT, the goal is to construct an instance (G, k) of IS so that Φ is satisfiable if and only if (iff) G has an independent set of size k.

PS: It suffices to show for *exactly* k, not $\geq k$.

The Polynomial-time Reduction

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

$$\downarrow\uparrow$$

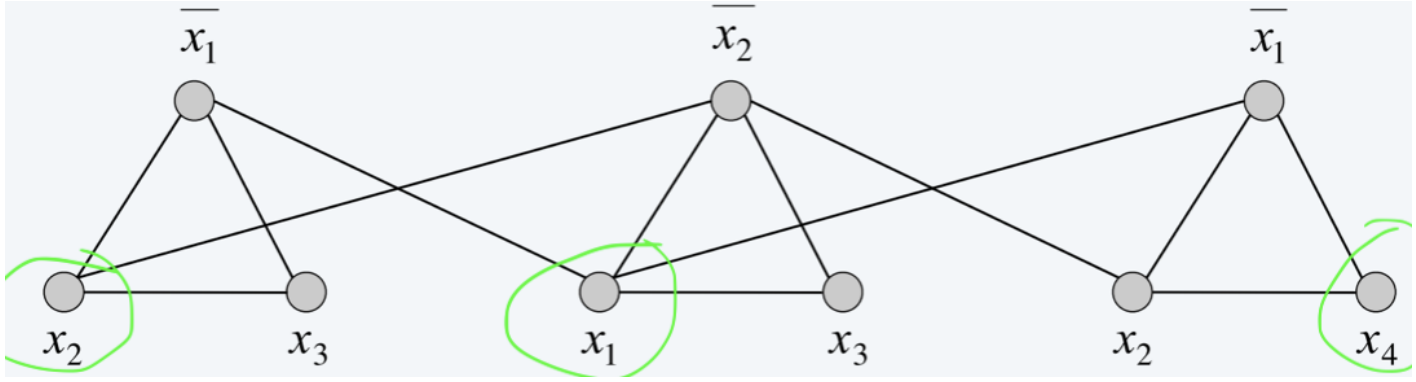


Do you see the required (polynomial-time) steps?

Φ is a YES (3-SAT) $\rightarrow$ (G, k) is a YES (IS)

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

$\downarrow$

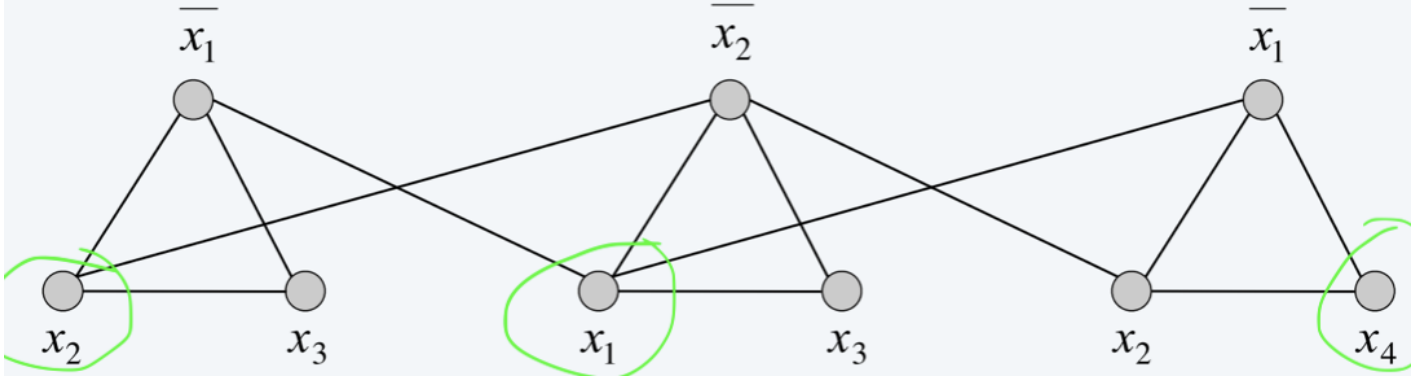


Suppose Φ is a YES-instance of 3-SAT,
then (G, k) is a YES-instance of IS.

Often forgotten (iff): Φ (3-SAT) $\leftarrow$ (G, k) (IS)

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

$\uparrow$



Suppose (G, k) is a YES-instance of IS,
then Φ is a YES-instance of 3-SAT.

Worst-case Analysis

The proof that we have just seen shows that *some instances* of IS are as hard to solve as the 3-SAT problem.

This does not mean that all instances of the IS problem are hard!

So, if there is no polynomial-time algorithm that solves 3-SAT on *all* instances, there is no polynomial-time algorithm that solves IS on *all* instances.

Question 3 at VisuAlgo Online Quiz

What is the best way to solve `IS` if the given graph G is an (unweighted) *Tree*?

A). Complete Search (Brute Force), `IS` is NP-complete

B). Divide and Conquer

C). Greedy Algorithm

D). Dynamic Programming

E). Randomized Algorithm

F). I do not know... something outside CS3230 syllabus?

Prove Vertex-Cover (VC) is NP-complete

Just now, we have seen that IS is NP-complete.

We also know, from the previous lecture, that $\texttt{IS} \leq_p \texttt{VC}$.

So VC is also NP-complete
(don't forget to quickly show that VC is in NP).

Side Note: Status of 3-SAT

The fastest algorithm known for 3-SAT runs in time $\approx 1.308^n$.

It is believed that there is no $2^{o(n)}$-time algorithm for 3-SAT (Exponential Time Hypothesis).

As we have seen with 3-SAT $\leq_p$ IS, it is often very convenient to reduce from 3-SAT to other problems in order to show that those other problems will also be hard if 3-SAT is hard.

See the out-degree of 3-SAT at `https://visualgo.net/en/reductions`.

Hitting-Set (HS)

Read https://visualgo.net/en/reductions?slide=19.

Show that the Hitting-Set Problem is NP-complete.
Hints: Try a reduction from Vertex-Cover.

Hitting-Set is in NP

A hitting set of size at most k can act as a certificate.

The size of this certificate is polynomial.

It is easy to verify this certificate in polynomial-time
(by checking its size is $\leq k$
and whether it has a non-empty intersection with all S_i).

VC $\leq_p$ HS

Read https://visualgo.net/en/reductions?slide=31 and https://visualgo.net/en/reductions?slide=31-1.

Impact?

Garey and Johnson's book, "Computers and Intractability", includes over 300 NP-complete problems and is the number 1 cited reference in computer science!

NP-completeness is used in more than 6000 publications per year (more than 'compiler', 'OS', 'database').

Main intellectual export of Computer Science.

ADS: Computational Complexity Course(s)

There are problems that are provably harder than NP-complete problems, problems that require polynomial space, problems that require large circuits, problems that are unsolvable even with unlimited time!

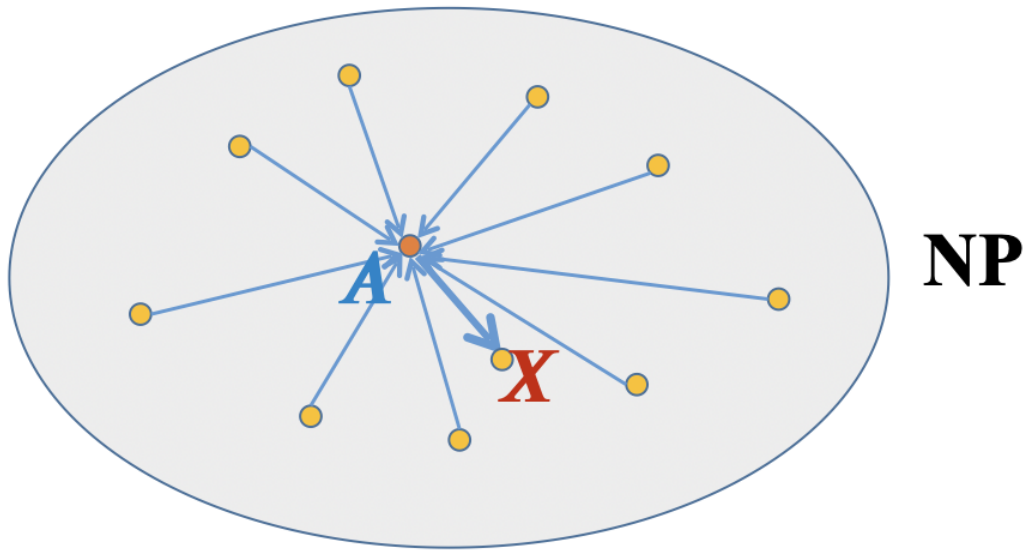
Enter the world of complexity theory...

ADS: Take Computational Complexity courses (CS3231, CS5230)!

Recap of this lecture (1)

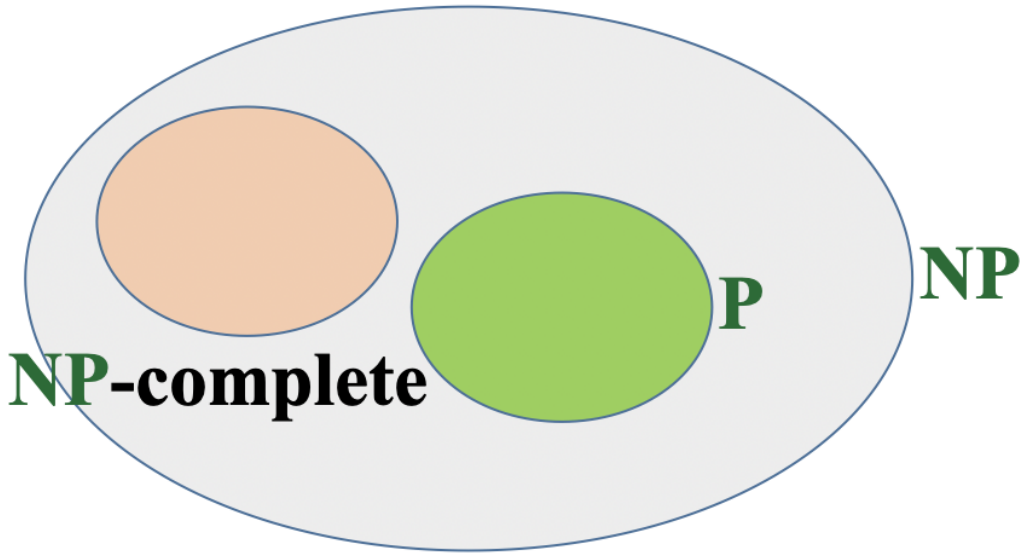
How to show a problem to be in NP-complete?
Let X be a problem which we wish to show to be NP-complete

1. Show that $X \in$ NP.
2. Pick a problem A which is already known to be NP-complete; Then, show that $A \leq_p X$, i.e., X is NP-hard.



Recap of this lecture (2)

Recall: A problem X in NP class is in NP-complete if **for every** $A \in$ **NP,** $A \leq_p X$.



Implication: If *any* NP-complete problem is solved in polynomial-time, then P = NP.

Problems in NP but believed not to be NP-complete

There are some problems in NP but believed not to be NP-complete, e.g., `Integer-Factoring`:
Given an integer, compute all its prime factors.
Decision version: Given an integer n and two integers $2 \leq d_1 < d_2 < n$, is there any prime factor of n in $[d_1, d_2]$?

It belongs to NP: Given any integer $x \leq d$,
it is possible, in polynomial-time, to determine if x is a prime and to determine if x divides n.

`Integer-Factoring` is believed to be *more difficult* than the problems in P, and *easier* than the problems in NP-complete.
There is no proof exists till now and the research on this continues.

ADS: Optimisation Algorithms Course (CS4234)

What to do when a problem is NP-complete?

Unless P=NP,
NP-complete problems have no poly-time algorithms.

But they come up frequently in real-life, so what can we do?

- We can try to solve *smaller* instances optimally using exponential time algorithms (brute force or cleverer methods such as branch and bound).
- We can check if the problem instance has special features that make it more efficiently solvable, e.g., if it is a 0/1-Knapsack problem with a *small capacity* W, then we can use the pseudo-polynomial DP algorithm.
- We can design an *approximation* or *local search* algorithm

Acknowledgement

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