MA1521: Cheat Sheet

Functions & Maximal Domains

- **Common Domains:**

Terms	Conditions
$\sqrt{g(x)}$	$g(x) \geq 0$
$\ln(g(x))$	$g(x) > 0$
$\frac{1}{g(x)}$	$g(x) \neq 0$
$\sin^{-1}(g(x))$	$[-1, 1]$

- **Composite:** $(f \circ g)(x) = f(g(x))$ (Note: $R_g \subseteq D_f$)
- **One-one:** **No** $x_1, x_2 \in D_f$ where $f(x_1) = f(x_2)$ Prove using **Horizontal Line Test**
- **Inverse: Only if one-one!** Reflect along $y = x$ incl. asymptotes! (i.e. $(a, b) \to (b, a)$). To find $f^{-1}(x)$, let $y = f^{-1}(x)$ solve for $x = f(y)$

Limits and Continuity

Let $f(x) = \begin{cases} g(x) & x < c \\ \alpha & x = c \\ h(x) & x > c \end{cases}$

- Left lim: $\lim_{x\to c-} f(x) = \lim_{x\to c-} g(x)$
- Right lim: $\lim_{x\to c+} f(x) = \lim_{x\to c+} g(x)$
- **Common Limit:** $\lim_{x\to c} f(x)$ if $\lim_{x\to c-} f(x) = \lim_{x\to c+} f(x) = L \in \mathbb{R}$
- f is **continuous at** $x = c$ if: Common Limit L at $x = c$ and $f(c) = L$
- If f & g continuous at $x = c$, these **are also continuous**: $f + g$, cf, $f \times g$, f/g

Laws of Limits

- $\lim_{x\to c}(f(x) \pm g(x)) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$
- $\lim_{x\to c}(\alpha f(x)) = \alpha \lim_{x\to c} f(x), \alpha$ is constant
- $\lim_{x\to c}(f(x) \cdot g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$
- $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}, \lim_{x\to c} g(x) \neq 0$
- For $|x|$, check value close to $x = c$
- g continuous at $x = c$ and $\lim_{x\to c} f(x) = c$, then $\lim_{x\to c} g(f(x)) = g(c) = g(\lim_{x\to c} f(x))$

Limits at Infinity: $x \to \infty$

- $\lim_{x\to+\infty} f(x) = c$ or $\lim_{x\to-\infty} f(x) = c$ implies $y = c$ is horizontal asymptote
- $\lim_{x\to\infty} \frac{1}{x^k} = 0$ for $k \in \mathbb{R}^+$
- $\lim_{x\to\infty} e^{-x} = 0$ & $\lim_{x\to-\infty} e^{x} = 0$
- $$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \frac{Ax^\alpha + \ldots}{Bx^\beta + \ldots} = \begin{cases} 0 & \alpha < \beta \\ \frac{A}{B} & \alpha = \beta \\ \underbrace{\infty / -\infty}_{\text{depends on qn}} & \alpha > \beta \end{cases}$$

  where α, β is the **highest power** (**NOTE:** only for ∞ limit!) [add lg vs 2^N]

Indeterminate Forms

- Indeterminate forms are of type: $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty, 0^0, 1^\infty$ & ∞^0
- For $\boldsymbol{x \not\to \infty}$ limit of $\frac{\mathbf{0}}{\mathbf{0}}$ or $\frac{\boldsymbol{\infty}}{\boldsymbol{\infty}}$ type, do one of the following:
  - Factorise the terms and cancel out:
    $$\frac{x^2 + 3x + 2}{1 - x^2} = \frac{\cancel{(x+1)}(x+2)}{(1-x)\cancel{(x+1)}}$$
  - Use for $\sqrt{\boldsymbol{x}} \pm \sqrt{\boldsymbol{x}}$ type:
    $$\sqrt{a} \pm \sqrt{b} = \frac{a \pm b}{\sqrt{a} \mp \sqrt{b}}$$
- If $\lim_{x\to c} g(x) = 0$ then:
  - $\lim_{x\to c} \frac{\sin g(x)}{g(x)} = \lim_{x\to c} \frac{g(x)}{\sin g(x)} = 1$
  - $\lim_{x\to c} \frac{\tan g(x)}{g(x)} = \lim_{x\to c} \frac{g(x)}{\tan g(x)} = 1$
- For $\mathbf{0^0, 1^\infty, \infty^0}$ use formula: $\lim_{x\to c}(f(x))^{g(x)} = \exp(\lim_{x\to c}(g(x) \cdot \ln f(x)))$
- **L'Hôpital Rule**:
  - $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$
  - For $0 \cdot \infty, \infty - \infty$, express to $\frac{0}{0}, \frac{\infty}{\infty}$
  - Not for $\cot, \operatorname{cosec}$ type (i.e. complex $f(x)$) or requiring repeated application

Squeeze Theorem

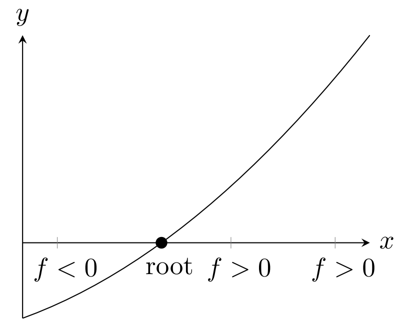
- **Thm I** For $f(x) \leq g(x) \leq h(x)$ if $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$ $\implies \lim_{x\to c} g(x) = L$
- **Thm II** For $\lim_{x\to c} f(x) = 0$ and $g(x)$ **is bounded** $\implies \lim_{x\to c} f(x)g(x) = 0$

  Note! If $g(x)$ unbounded ($g(x)$ can $= \pm\infty$), $\lim_{x\to c} f(x)g(x) = 0 \cdot \infty$, which is **indeterminate!**
  - Example $g(x)$: $|\sin a(x)| \leq 1$, $|\cos a(x)| \leq 1$, $|\sin a(x) \cdot \cos a(x)| \leq 1$

Intermediate Value Theorem

For $\boldsymbol{f}$ continuous on $[\boldsymbol{a}, \boldsymbol{b}]$

- $f(a) < k < f(b)$, $f(c) = k, c \in [a, b]$
- $f(a) \times f(b) < 0$, $f(x)$ has **at least one** real root
- Repeated IVT will allow us to approximate root by certain degree of accuracy (Bisection Method)



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Differentiability

- f differentiable at $x = x_0$ if lim exists

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$$

- if f differentiable at $x = x_0$, means f continuous at $x = x_0$

Implicit Differentiation

- **Method I:**

$$\frac{d}{dx}g(y) = g'(y)\cdot\frac{dy}{dx}$$

- **Method II:**
  Let $f_x(x,y)$ be partial derivative of $f(x,y)$ w.r.t. x, treating y as constant.

  Let $f_y(x,y)$ be partial derivative of $f(x,y)$ w.r.t. y, treating x as constant.

  Then, find: $\frac{dy}{dx} = -\frac{f_x}{f_y}$

Derivative of Inverse Functions

- Only if f is one-one and is differentiable on an interval I

- At point $(a, f^{-1}(a))$, where $a \in R_f$:

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

  provided $f'(f^{-1}(a)) \neq 0$

Parametric Equations

- $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, where $\frac{dx}{dt} \neq 0$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

- $$= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Differentiating Special Forms

- For $\boldsymbol{(f(x))^{g(x)}}$ form:

$$\frac{d}{dx}(f(x))^{g(x)}$$
$$= (f(x))^{g(x)}\times\left[g'(x)\ln f(x) + \frac{f'(x)}{f(x)}g(x)\right]$$

- For $\boldsymbol{\log_{g(x)} f(x)}$ form, change base:

$$\log_{g(x)} f(x) = \frac{\ln f(x)}{\ln g(x)}$$

  only when $f(x), g(x) > 0$ & $g(x) \neq 1$

Applications of Derivatives

- **Tangents & Normals**

  Tangent: $y - y_0 = m(x - x_0)$

  Normal: $y - y_0 = \frac{-1}{m}(x - x_0)$

- **Increasing & Decreasing $\boldsymbol{f(x)}$**
  Let f be continuous on $[a,b]$ & differentiable on (a,b)

  Increasing on $[a,b]$: $f' > 0$

  Decreasing on $[a,b]$: $f' < 0$

  $\forall x \in (a,b)$

- **Concavity of f(x)**
  Let f be twice differentiable on (a,b)

  Concave Up on (a,b): $f'' > 0$

  Concave Down on (a,b): $f'' < 0$

  $\forall x \in (a,b)$

Maximum & Minimum

- A function f defined over an interval I has
  - **absolute/global maximum** at c if $f(c) \geq f(x), \forall x \in I$
  - **absolute/global maximum** at c if $f(c) \leq f(x), \forall x \in I$
  - **relative/local maximum** at c in interval $J \subseteq I$ s.t. $f(c) \geq f(x), \forall x \in J$
  - **relative/local minimum** at c in interval $J \subseteq I$ s.t. $f(c) \leq f(x), \forall x \in J$
- **Extreme Value Thm**: If f is continuous on $[a,b]$, there's pts $c, d \in [a,b]$ s.t. f attains abs max at c and abs min at d
- **Critical Pt**: f over I has critical pt at $c \in I$ (ex. endpts), if $f'(c) = 0$ or d.n.e.

Absolute/Relative Extrema

- **Finding Absolute Extrema**:
  1. Record $f(x)$ at critical & end pts
  2. Pick largest and smallest $f(x)$ amongst values found in 1.
  3. If largest or smallest value, $c \notin D_f$, then $f(x)$ has no abs max/min (depending on the value)
- **Finding Relative Extrema**:
  1. Find all critical pts over interval I
  2. Use First Derivative Test:
     if $f'(x)$ changes from + to −
     $\implies$ **local max**

     if $f'(x)$ changes from − to +
     $\implies$ **local min**

     if $f'(x)$ no sign change
     $\implies$ inflexion pt (**NOT** local extrema)

Geometric Sequences/Series

- $\sum_{k=0}^{\infty} r^k = \sum_{k=1}^{\infty} r^{k-1} = \frac{1}{1-r}$ iff $|r| < 1$
- $\sum_{k=m}^{\infty} r^k = r^m \sum_{k=0}^{\infty} r^k (= r^m + r^{m+1} + r^{m+2} + ...)$

Telescoping Series

- $\sum_{k}^{\infty}(u_k - u_{k-1})$ or $\sum_{k}^{\infty}(u_{k-1} - u_k)$
- $\sum_{k=m}^{\infty}(u_k - u_{k-1}) = \lim_{N\to\infty}\sum_{k=m}^{N}(u_k - u_{k-1})$

Useful for Convergence Tests

- $(n+1)! = (n+1)\cdot n!$
- $\lim_{n\to\infty} n^{1/n} = 1$ & $\lim_{n\to\infty}\left(1 + \frac{y}{n}\right)^n = e^y$

Power Series

- $\sum_{n=0}^{\infty} c_n(x-a)^n$ is a power series centered at $x = a$. When $x = a$, series $= c_0$
- **Radius of convergence:**
  R s.t. series conv. when $|x - a| < R$ & div. when $> R$.
- **Finding $\boldsymbol{R}$:**
  $L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$, where $u_n = c_n(x-a)^n$

L	Conclusion	R
0	Conv. $\forall x$	∞
∞	Conv. only for $x = a$	0
$> 0, \neq \infty$	Conv. when $\lvert x - a\rvert < R$	$L < 1$

Taylor & Maclaurin Series

- $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is the Taylor series at $x = a$ (Maclaurin series: $a = 0$)
- $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} c_n(x-a)^n$ then: $c_n = \dfrac{f^{(n)}(a)}{n!} \to f^{(n)}(a) = n! \cdot c_n$

Maximal Domain of 3-variables

Partial Derivatives

- $\dfrac{\partial z}{\partial x} = \displaystyle\lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$
- $$\begin{aligned}\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} &= f_x(x,y) = f_1(x,y) \\ &= D_z f(x,y) = D_1 f(x,y)\end{aligned}$$

Higher Order Partial Derivatives

- $f_{xx} = (f_x)_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2}$
- $f_{xy} = (f_x)_y = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x}$
- $f_{yx} = (f_y)_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y}$
- $f_{yy} = (f_y)_y = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2}$

Tangents and Plane of $z = f(x,y)$

Tangents of Surface Intersections

- Tangent of plane $y = b$ intersection of surface, creating an x-curve $z = f(x,b)$
  $$\mathbf{r} = \begin{pmatrix} a \\ b \\ f(a,b) \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ f_x(a,b) \end{pmatrix}$$
- Tangent of plane $x = a$ intersection of surface, creating an y-curve $z = f(a,y)$
  $$\mathbf{r} = \begin{pmatrix} a \\ b \\ f(a,b) \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ f_y(a,b) \end{pmatrix}$$

Plane of Surfaces at point P

- Normal vector Π
  $$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \\ f_y(a,b) \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ f_x(a,b) \end{pmatrix} = \begin{pmatrix} f_x(a,b) \\ f_y(a,b) \\ -1 \end{pmatrix}$$
- Vector equation of Π
  $$\mathbf{r} \cdot \begin{pmatrix} f_x(a,b) \\ f_y(a,b) \\ -1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ f(a,b) \end{pmatrix} \cdot \begin{pmatrix} f_x(a,b) \\ f_y(a,b) \\ -1 \end{pmatrix}$$
- Cartesian equation of Π
  $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$
- Linear approximation of point on plane
  $f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Chain Rule

One Ind. & Two Dep. Variables

Let $F = f(x,y)$ and $x = x(t), y = y(t)$. Then,

$$\frac{dF}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Two Ind. & Two Dep. Variables

Let $F = f(x,y)$ and $x = x(s,t), y = y(s,t)$. Then,

$$\frac{\partial F}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial F}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Directional Derivatives

$$\begin{aligned} D_u f(a,b) &= f_x(a,b)u_1 + f_y(a,b)u_2 \\ &= \nabla f(a,b) \cdot \mathbf{u} \\ &= \begin{pmatrix} f_x(a,b) \\ f_y(a,b) \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{aligned}$$

in direction of unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$

Finding u when $D_u f(x,y)$ is 0

Consider $D_u f = 0$ and ${u_1}^2 + {u_2}^2 = 1$

Finding u when $D_u f(x,y)$ is max

Consider $\mathbf{u} = \widehat{\begin{pmatrix} f_x(a,b) \\ f_y(a,b) \end{pmatrix}}$

Optimisation

Critical Points

Point (a,b) is critical of f if $f_x(a,b) = f_y(a,b) = 0$ or either f_x and f_y d.n.e

Second Derivative Test

$$D = f_{xx}(a,b) \cdot f_{yy}(a,b) - (f_{xy}(a,b))^2$$

D	f_{xx}	Type
> 0	< 0	Local Max
> 0	> 0	Local Min
< 0	any	Saddle
$= 0$	any	Inconclusive

Lagrange Multiplier

Extrema of $f(x_1, x_2, \ldots, x_n)$ subject to constraint $g(x_1, x_2, \ldots, x_n) = c$.
We solve a system of $(n+1)$ equations:

$$\frac{\partial f}{\partial x_1} = \lambda \frac{\partial g}{\partial x_1}$$
$$\frac{\partial f}{\partial x_2} = \lambda \frac{\partial g}{\partial x_2}$$
$$\vdots$$
$$\frac{\partial f}{\partial x_n} = \lambda \frac{\partial g}{\partial x_n}$$

Fundamental Theorem of Calculus

FTC1 Definite Integral

FTC2

$$\begin{aligned} &\frac{d}{dx} \int_{v(x)}^{u(x)} f(t)dt \\ &= f(u(x)) \cdot u'(x) - f(v(x)) \cdot v'(x) \end{aligned}$$

L'Hôpital Rule using FTC

$$\begin{aligned} &\lim_{x\to\infty} \left(\frac{\int_a^b (f(t))dt}{\int_c^d (g(t))dt} \right) \\ &= \lim_{x\to\infty} \left(\frac{\frac{d}{dx} \int_a^b (f(t))dt}{\frac{d}{dx} \int_c^d (g(t))dt} \right) \end{aligned}$$

Area Between Curves

Area bounded by curves $y = f(x)$ and $y = g(x)$ and lines $x = a, x = b$ is

$$\int_a^b |f(x) - g(x)| dx$$

Volume of Solids of Revolution

Disc Method Volume of solid generated by rotating the area bounded by $y = f(x)$, $y = k$ and $x = a, x = b$ along x-axis is

$$V = \pi \int_a^b (y-k)^2 dx$$

Volume of solid generated by rotating the area bounded between $y = f(x), y = g(x)$, where $f(x) \geq g(x)$, and $x = a, x = b$ along x-axis is

$$V = \pi \int_a^b ((f(x))^2 - (g(x))^2) dx$$

Shell Method Volume of solid generated by rotating the area bounded by $y = f(x)$, $y = k$ and $x = a, x = b$ along y-axis is

$$V = 2\pi \int_a^b x|f(x) - k| dx$$

Volume of solid generated by rotating the area bounded between $y = f(x), y = g(x)$, where $f(x) \geq g(x)$, and $x = a, x = b$ along y-axis is

$$V = 2\pi \int_a^b x|f(x) - g(x)| dx$$

Reduction Formulae

Let I_n be an integral. Question will ask to show $I_n = const - nI_{n-1}$

First Order Ordinary D.E.

Separable ODE

In the form: $\frac{dy}{dx} = f(y)g(x)$

Solve for: $\int \frac{1}{f(y)} dy = \int g(x) dx$

Linear First Order ODE

In the form: $\frac{dy}{dx} + P(x)y = Q(x)$

Let $I(x) = e^{\int P(x)dx}$ (integrating factor).

$$\frac{d}{dx}(y \cdot I(x)) = I(x)Q(x)$$
$$y \cdot I(x) = \int I(x)Q(x) dx$$

Bernoulli DE

In the form: $\frac{dy}{dx} + P(x)y = Q(x)y^n, n \neq 1$

Substitute $z = y^{1-n}$, reducing DE to

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$$

and solve using Linear First Order ODE method (integrating factor)

Second Order Ordinary D.E.

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = d(x)$$

Homogeneous

Condition: $d(x) = 0$

Let α, β be roots of the following **auxiliary equation**

$$am^2 + bm + c = 0$$

Case	General Solution
α, β distinct	$y = Ae^{\alpha x} + Be^{\beta x}$
$\alpha = \beta$	$y = (Ax + B)e^{\alpha x}$
α, β imag	$y = e^{px}(A\cos qx + B\sin qx)$ $\alpha, \beta = p \pm iq$

Non-Homogeneous

Condition: $d(x) \neq 0$

Solve corresponding homogeneous DE, general solution is

$$y(x) = Ay_1(x) + By_2(x) \quad \text{(H)}$$

Find **particular solution** of equation in form:

$$y(x) = u_1(x) \cdot y_1(x) + u_2(x) \cdot y_2(x) \quad \text{(NH)}$$

$u_1(x)$ and $u_2(x)$ are as follows:

$$u_1(x) = \frac{1}{a} \int \frac{-y_2(x) \cdot d(x)}{W(y_1, y_2)} dx$$
$$u_2(x) = \frac{1}{a} \int \frac{y_1(x) \cdot d(x)}{W(y_1, y_2)} dx$$

where $W(y_1, y_2) = y_1(x) \cdot y_2'(x) - y_2(x) \cdot y_1'(x)$

Hence, general solution of D.E. is:

$$y = \underbrace{Ay_1(x) + By_2(x)}_{\text{GS of H}} + \underbrace{u_1(x) \cdot y_1(x) + u_2(x) \cdot y_2(x)}_{\text{PS of NH}}$$

Newton-Raphson Iteration

- Find init. estimate α_0 using IVT (in between the two values determined)
- For n = 0,1,2,...

$$\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)}$$

Trapezoidal Rule

$$J = \int_a^b f(x)dx \approx \frac{h}{2}[f_0 + 2(f_1 + \ldots + f_{n-1}) + f_n]$$

where $h = \frac{b-a}{n}$, n is no. of trapezia ($n-1$ ordinates)

