CS1231: Reference

Proof Language

- $\exists!$: Exists a unique...
- $\equiv$: Logical Equivalence: identical truth table (Definition 2.1.6)

Order of Operations

- $\neg$, followed by $\wedge\vee$, followed by $\rightarrow\leftrightarrow$

Proving Methods

- **Construction**: just sub in all $x \in D$
- **Counterexample**: show one condition that leads to contradiction
- **Contraposition**: To prove $P \rightarrow Q$, prove $\neg Q \rightarrow \neg P$
- **Contradiction**: To prove A, prove $\neg A$ is not true (Clearly this is absurd)

Thm 2.1.1 Logical Equivalences

(Aaron, pp 21 - 22)

- **Commutative Law**: $p \wedge q \equiv q \wedge p$
- **Associative Law**: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ (same with $\vee$)
- **Distributive Law**: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ (swap $\vee$ & $\wedge$)
- **Identity Law**: $p \wedge \mathbf{t} \equiv p$ or $p \vee \mathbf{c} \equiv p$
- **Negation Law**: $p \vee \neg p \equiv \mathbf{t}$ or $p \wedge \neg p \equiv \mathbf{c}$
- **Double Negation Law**: $\neg(\neg p) \equiv p$
- **Idempotent Law**: $p \wedge p \equiv p$ (same for $\vee$)
- **Universal Bound Law**: $p \vee \mathbf{t} \equiv \mathbf{t}$ or $p \wedge \mathbf{c} \equiv \mathbf{c}$
- **De Morgan's laws**: $\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Conditional Statements

- $p \rightarrow q \equiv \neg p \vee q$
- **Contrapositive** (Def 2.2.2): $p \rightarrow q \equiv \neg q \rightarrow \neg p$
- **Converse** (2.2.3): $q \rightarrow p$
- **Inverse** (2.2.4): $\neg p \rightarrow \neg q$
- $p \rightarrow q \not\equiv (q \rightarrow p \equiv \neg p \rightarrow \neg q)$
- **Only If** (2.2.5): p only if $q \equiv p \rightarrow q$
- **Biconditional** (2.2.6): $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
- **Necessary & Sufficient Cond** (2.2.7): r necessary $s \equiv r \rightarrow s$ r sufficient $s \equiv \neg r \rightarrow \neg s$

Valid Arguments

- **Argument** (2.3.1): If all premise true, conclusion must be true
- **Syllogism**: Two premises and one conclusion
- **Modus Ponens**: $p \rightarrow q$, p, $\therefore q$
- **Modus Tollens**: $p \rightarrow q$, $\neg q$, $\therefore \neg p$

Rules of Inference

- **Generalization**: $p, \therefore p \vee q$
- **Specialization**: $p \wedge q, \therefore p$
- **Elimination**: $p \vee q$, $\neg p$, $\therefore q$
- **Transitivity**: $p \rightarrow q$, $q \rightarrow r$, $\therefore p \rightarrow r$
- **Proof by Division into Cases**: $p \vee q$, $p \rightarrow r$, $q \rightarrow r$, $\therefore r$

Rules of Inference (Wei Quan)

- **Conjunction Intro** – $A, B, \therefore A \wedge B$
- **Conjunction Elim** – $A \wedge B, \therefore A, B$
- **Disjunction Intro** – $A, \therefore A \vee B, B \vee A$
- **Disjunction Elim** : $A \vee B, A \rightarrow C, B \rightarrow C, \therefore C$
- **Contradiction Intro** – $A, \neg A, \therefore Cont.$
- **Contradiction Elim**: $A \rightarrow$ Contradiction, $\therefore \neg A$
- **Double Negation Elim** – $\neg\neg A, \therefore A$

Fallacies

- Converse Error: $p \rightarrow q$, q, $\therefore p$
- Inverse Error: $p \rightarrow q$, $\neg p$, $\therefore \neg q$
- Sound & Unsound Argument: Sound iff valid and premises are true

Predicates & Quantified Stmt

- **Predicate** (3.1.1): A **predicate** sentence contains a finite number of variables and becomes a stmt when specific values are subst in the vars. The **domain** of a predicate var is the set of all values that may be subst in place of the var.
- **Truth Set** (3.1.2): If P(x) is a predicate and $D_x \equiv D$, the truth set is the set of all elements of D that make $P(x)$ true when they are subst for x. The truth set of $P(x)$ is $\{x \in D | P(x)\}$.
- **Universal Stmt** (3.1.3): $\forall x \in D, Q(x)$
  - Equivalent to $Q(x_1) \wedge Q(x_2) \wedge ... \wedge Q(x_n)$
  - Stmt is true iff $Q(x)$ true $\forall x \in D$
  - Stmt is false iff $Q(x)$ false for at least one $x \in D$
- **Existential Stmt** (3.1.4): $\exists x \in D$, such that $Q(x)$
  - Equivlent to $Q(x_1) \vee Q(x_2) \vee ... \vee Q(x_n)$
  - Stmt is true iff $Q(x)$ true for at least one $x \in D$
  - Stmt is false iff $Q(x)$ false $\forall x \in D$
- **Implicit Quantification**: $\implies$ $\iff$
  - $P(x) \implies Q(x)$: truth set $P(x) \subset$ truth set $Q(x)$
  - $P(x) \iff Q(x)$: truth set $P(x) \equiv$ truth set $Q(x)$

Negation of Quantified Stmt

- **Negation of Universal Stmt** (Thm 3.2.1)

  $\sim (\forall x \in D, P(x)) \equiv \exists x \in D,$ s.t. $\sim P(x)$
- **Negation of Existential Stmt** (Thm 3.2.1)

  $\sim (\exists x \in D,$ s.t. $P(x)) \equiv \forall x \in D, \sim P(x)$

Universal Conditional Stmt

$$\forall x \in D, P(x) \implies Q(x)$$

- **Vacuously True**: iff $P(x)$ false $\forall x \in D$
- **Contrapositive**: $\forall x \in D, \sim Q(x) \implies \sim P(x)$
- **Converse**: $\forall x \in D, Q(x) \implies P(x)$
- **Inverse**: $\forall x \in D, \sim P(x) \implies \sim Q(x)$
- Refer to 2.2.5 and 2.2.7 for only if, necessary & sufficient conditions
- **Universal Modus Ponens & Tollens**: $\forall x \in D, P(x) \implies Q(x), P(a)$ for $a \in D$ $\therefore Q(a)$ ($\sim Q(a), \therefore \sim P(a)$ for tollens)

CS1231: Number Theory

Def/Thm in Lecture Slides

- **Even & Odd** (Def 1.6.1, Proofs Handout, TS):
$$n \text{ is even} \iff \exists k \in \mathbb{Z} \text{ such that } n = 2k$$
$$n \text{ is odd} \iff \exists k \in \mathbb{Z} \text{ such that } n = 2k+1$$
- **Divisibility** (Def 1.3.1, PH):
$d \mid n \iff \exists k \in \mathbb{Z}$, s.t. $n = dk$
- **Thm 4.1.1** (pg 4, Number Theory Week 4, TS) :
$\forall a, b, c \in \mathbb{Z}$, if $a \mid b$ & $a \mid c$, then $\forall x, y \in \mathbb{Z}, a \mid (bx + cy)$
- **Prop 4.2.2** (p9, NTW4, TS):
For any two primes p and p', if $p \mid p'$ then $p = p'$
- **Thm 4.2.3** (pg 16, NTW4):
If p is prime and $x_1, x_2, ..., x_n$ are any integers s.t.:
$$p \mid x_1 x_2 ... x_n,$$
then $p \mid x_i$ for some $x_i (1 \leq i \leq n)$
- **Lower Bound** (Def 4.3.1, NTP2, p3):
$b \in \mathbb{Z}$ is lower bound for set $X \subseteq \mathbb{Z}$ if $b \leq x, \forall x \in X$
- Well Ordering Principle (Thm 4.3.2, NTP2, p5):
If non-empty set $S \subseteq \mathbb{Z}$ has lower/upper bound, then S has a least/greatest element
- Uniqueness of least element (Prop 4.3.3, NTP2, p8):
If set $S \subseteq \mathbb{Z}$ has least/greatest element, then least/greatest elem is unique
- Quotient-Remainder Thm (Thm 4.4.1):
Given any $a \in \mathbb{Z}$ & any $b \in \mathbb{Z}^+, \exists! q, r \in \mathbb{Z}$ s.t.:
$$a = bq + r \ \& \ 0 \leq r < b$$
- G.C.D. (Def 4.5.1, NTP2, p21):
Let $a, b \in \mathbb{Z}$, not both zero, g.c.d. of a, b, $\gcd(a, b)$, is $d \in \mathbb{Z}$ satisfying:
$$d \mid a \ \& \ d \mid b \tag{1}$$
$$\forall c \in \mathbb{Z}, \text{if } c \mid a \ \& \ c \mid b \text{ then } c \leq d \tag{2}$$
- Existence of gcd (Prop 4.5.2):
For any $a, b \in \mathbb{Z}$, not both zero, their gcd exists and unique
- Bézout's Identity (Thm 4.5.3):
Let $a, b \in \mathbb{Z}$, not both zero, & $d = \gcd(a, b)$.
Then, $\exists x, y \in \mathbb{Z}$ s.t.:
$$ax + by = d$$
- Relatively Prime/Coprime (Def 4.5.4):
$a, b \in \mathbb{Z}$ are coprime $\iff \gcd(a, b) = 1$
- Prop 4.5.5:
$a, b \in \mathbb{Z}$, not both zero, if c is common divisor of a & b, then $c \mid \gcd(a, b)$
- NTP2, p38: $\forall a, b \in \mathbb{Z}^+, a \mid b \iff \gcd(a, b) = a$
- L.C.M. (Def 4.6.1, NTP2, p41):
$a, b \in \mathbb{Z} \setminus \{0\}$, their l.c.m, denoted $\text{lcm}(a, b)$, is $m \in \mathbb{Z}^+$ s.t.
$$a \mid m \ \& \ b \mid m \tag{3}$$
$$\forall c \in \mathbb{Z}^+, \text{if } a \mid c \ \& \ b \mid c, \text{then } m \leq c \tag{4}$$
- NTP2, p43: $\forall a, b \in \mathbb{Z}^+, \gcd(a, b) \mid \text{lcm}(a, b)$

Theorems By Epp

- Thm 4.3.1: $\forall a, b \in \mathbb{Z}^+, \text{if } a \mid b \text{ then } a \leq b$
- Thm 4.3.3: $\forall a, b, c \in \mathbb{Z}, \text{if } a \mid b \text{ and } b \mid c \text{ then } a \mid c$
- Thm 4.3.5: Given any integer $n > 1$, $\exists k \in \mathbb{Z}^+$, distinct primes $p_1, p_2, ..., p_k$ & positive integers $e_1, e_2, ..., e_k$, s.t.
$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} ... p_k^{e_k}$$
and any other exp for n as a product of prime numbers is identical to this (except ordering)
- Thm 4.7.1: $\sqrt{2}$ is irrational
- Prop 4.7.3:
For any $a \in \mathbb{Z}$ and any prime p, if $p \mid a$ then $p \nmid (a + 1)$
- Thm 4.7.4: The set of primes is infinite

Appendix A (Epp)

- T1 (Cancellation Law for Add): $a + b = a + c \implies b = c$
- T2 (Possibility of Subtraction): Given a, b, $\exists! x$ such that $a + x = b$. This x is denoted by $b - a$.
- T3: $b - a = b + (-a)$ T4: $-(-a) = a$
- T5: $a(b - c) = ab - ac$ T6: $0 \cdot a = a \cdot 0 = 0$
- T7 (Cancellation Law for Multiplication):
$ab = ac, a \neq 0 \implies b = c$
- T8 (Possibility of Division):
Given a, b with $a \neq 0$, $\exists! x$ such that $ax = b$. This x is denoted b/a and is called the **quotient** of b and a
- T9: $a \neq 0 \implies b/a = b \cdot a^{-1}$
- T10: $a \neq 0 \implies (a^{-1})^{-1} = a$
- T11 (Zero Product Property): $ab = 0 \implies a \lor b = 0$
- T12 (Rule for Multiplication with Negative Signs):
$(-a)b = a(-b) = -(ab)$, $(-a)(-b) = ab$
& $-\dfrac{a}{b} = \dfrac{-a}{b} = \dfrac{a}{-b}$
- T13 (Equivalent Fractions Property): $\dfrac{a}{b} = \dfrac{ac}{bc}; b, c \neq 0$
- T14 (Rule for Addition of Fractions): $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}; b, d \neq 0$
- T15 (Rule for Multiplication of Fractions):
$\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}, b \neq 0, d \neq 0$
- T16 (Rule for Division of Fractions):
$\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{ad}{bc}, b \neq 0, c \neq 0, d \neq 0$
- T17 (Trichotomy Law): $a < b, b < a$ or $a = b, \forall a, b \in \mathbb{R}$
- T18 (Transitive Law): $a < b, b < c \implies a < c$
- T19: $a < b \implies a + c < b + c$

CS1231: Number Theories

Basics

- **Even & Odd** (Def 1.6.1, Proofs Handout, TS):

$$\begin{aligned} n \text{ is even} &\iff \exists k \in \mathbb{Z} \text{ such that } n = 2k \\ n \text{ is odd} &\iff \exists k \in \mathbb{Z} \text{ such that } n = 2k+1 \end{aligned}$$

- **The sum of two even $\mathbb{Z}$ is even** (Thm 4.1.1, Epp)
- **Rational Number**
$r \in \mathbb{Q} \iff \exists a, b \in \mathbb{Z}, r = \frac{a}{b}$ & $b \neq 0$
- **Every $\mathbb{Z}$ is a rational number** (Thm 4.2.1, Epp)
- **The sum of any two rational numbers is rational** (Thm 4.2.2, Epp)
- **The double of a rational number is rational** (Col 4.2.3, Epp)

Divisibility

- **Divisibility** (Def 1.3.1, PH):
$d \mid n \iff \exists k \in \mathbb{Z}$, s.t. $n = dk$
- **Thm 4.1.1** (pg 4, Number Theory Week 4, TS) :
$\forall a, b, c \in \mathbb{Z}$, if $a \mid b$ & $a \mid c$, then $\forall x, y \in \mathbb{Z}, a \mid (bx + cy)$
- **Thm 4.3.1** (Epp): $\forall a, b \in \mathbb{Z}^+$, if $a \mid b$ then $a \leq b$
- **Thm 4.3.2** (Epp): $d \mid 1$, d is only $1, -1$
- **Thm 4.3.3** (Epp): $\forall a, b, c \in \mathbb{Z}$, if $a \mid b$ & $b \mid c$ then $a \mid c$
- **Thm 4.3.4** (Epp): Any integer $n > 1$ is divisible by a prime number

Prime Numbers

- **Definition of Prime**
$n \in \mathbb{Z}$ & $n > 1$ then,

$$\begin{aligned} n \text{ is prime} &\leftrightarrow \forall r, s \in \mathbb{Z}^+, n = rs \rightarrow \\ &\quad ((r = 1 \wedge s = n) \vee (r = n \wedge s = 1)) \\ n \text{ is composite} &\leftrightarrow \exists r, s \in \mathbb{Z}^+ \text{ s.t.} \\ &\quad (n = rs) \wedge (1 < r < n \wedge 1 < s < n) \end{aligned}$$

- **Proposition 4.2.2** (NTW4)
For any two primes p & p', if $p \mid p'$ then $p = p'$
- **Proposition 4.7.3** (Epp)
For any $a \in \mathbb{Z}$ and any prime p, if $p \mid a$ then $p \nmid (a + 1)$
- **The set of primes is infinite** (Thm 4.7.4, Epp)
- **Theorem 4.2.3** (pg 16, NTW4):
If p is prime and $x_1, x_2, ..., x_n$ are any integers s.t.:

$$p \mid x_1 x_2 ... x_n,$$

then $p \mid x_i$ for some $x_i (1 \leq i \leq n)$

- **Unique Prime Factorization** (Thm 4.3.5, Epp):
Given any integer $n > 1$, $\exists k \in \mathbb{Z}^+$, distinct primes $p_1, p_2, ..., p_k$ & positive integers $e_1, e_2, ..., e_k$, s.t.

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} ... p_k^{e_k}$$

and any other exp for n as a product of prime numbers is identical to this (except ordering)

Well Ordering Principle

- **Lower Bound** (Def 4.3.1, NTP2):
An integer b is said to be a **lower bound** for a set $X \subseteq \mathbb{Z}$ if $b \leq x$ for all $x \in X$
- **Well Ordering Principle** (Thm 4.3.2, NTP2):
If a non-empty set $S \subseteq \mathbb{Z}$ has a lower/upper bound, then S has a least/greatest element
- **Uniqueness of least element** (Prop 4.3.3, NTP2):
If a set S of integers has a least/greatest element, then the least/greatest element is unique

Quotient-Remainder Theorem

- **Quotient-Remainder Theorem** (Thm 4.4.1, NTP2):
Given any $a \in \mathbb{Z}$ and any $b \in \mathbb{Z}^+$, there exist unique integers q, r such that:

$$a = bq + r \text{ and } 0 \leq r < b$$

GCD/LCM

- **Greatest Common Divisor** (Def 4.5.1, NTP2):
Let a, b be integers, not both zero. The **greatest common divisor** of a and b, denoted $\gcd(a, b)$, is the integer d satisfying:

    (i) $d \mid a$ and $d \mid b$
    (ii) $\forall c \in \mathbb{Z}$, if $c \mid a$ and $c \mid b$, then $c \leq d$

- **Existence of gcd** (Prop 4.5.2, NTP2):
For any integers a, b, not both zero, their gcd exists and is unique
- **Bézout's Identity** (Thm 4.5.3, NTP2):
Let $a, b \in \mathbb{Z}$, not both zero, and let $d = \gcd(a, b)$. Then, there exists $x, y \in \mathbb{Z}$ such that: $ax + by = d$
- **Relatively Prime/Coprime** (Def 4.5.4, NTP2):
Integers a, b are **(relatively prime)/coprime** iff $\gcd(a, b) = 1$
- **Proposition 4.5.5** (NTP2):
For any $a, b \in \mathbb{Z}$, not both zero, if c is a common divisor of a and b then $c \mid \gcd(a, b)$
- **Some Theorem** (NTP2):
$\forall a, b \in \mathbb{Z}^+, a \mid b \iff \gcd(a, b) = a$
- **Theorem in Assignment 1**:
$\forall a, b \in \mathbb{Z}$, not both zero & $d = \gcd(a, b)$, then $\frac{a}{d}, \frac{b}{d} \in \mathbb{Z}$ with no common divisor that is greater than 1
- **Least Common Multiple** (Def 4.6.1, NTP2):
For any non-zero integers a, b, their **least common multiple**, denoted $\text{lcm}(a, b)$, is the positive integer m such that:

    (i) $a \mid m$ and $b \mid m$
    (ii) $\forall c \in \mathbb{Z}^+$, if $a \mid c$ and $b \mid c$, then $m \leq c$

- **Some other Theorem** (NTP2, last page):
$\forall a, b \in \mathbb{Z}^+, \gcd(a, b) \mid \text{lcm}(a, b)$

Modulo Arithmetic

- **Congruence Modulo** (4.7.1, NTP3):
  Let $m, n \in \mathbb{Z}$ & $d \in \mathbb{Z}^+$. m is congruent to n modulo d:
  $$m \equiv n \pmod d \iff d \mid (m-n)$$
- **Modular Equivalences** (8.4.1, Epp):
  For $a, b, n \in \mathbb{Z}, n > 1$. Then the following are *equivalent*:
  1. $n \mid (a-b)$
  2. $a \equiv b \pmod n$
  3. $a = b + kn, k \in \mathbb{Z}$
  4. a, b have same (non-negative) remainder when divided by n
  5. $a \mod n = b \mod n$
- **Modulo Arithmetic** (8.4.3, Epp):
  Let $a, b, c, d, n \in \mathbb{Z}$, $n > 1$ and suppose:
  $$a \equiv c \pmod n \ \& \ b \equiv d \pmod n$$
  Then
  1. $(a \pm b) \equiv (c \pm d) \pmod n$
  2. $ab \equiv cd \pmod n$
  3. $a^m \equiv c^m \pmod n, \forall m \in \mathbb{Z}^+$
- **Corollary 8.4.4** (Epp):
  For $a, b, c \in \mathbb{Z}, n > 1$. Then,
  $$ab \equiv [(a \mod n)(b \mod n)] \pmod n$$
  If $m \in \mathbb{Z}^+$, then,
  $$a^m \equiv [(a \mod n)^m] \pmod n$$
- **Multiplicative inverse modulo *n*** (4.7.2, NTP3):
  For $a, n \in \mathbb{Z}, n > 1$, if $s \in \mathbb{Z}, as \equiv 1 \pmod n$, then s is the **multiplicative inverse of *a* modulo *n***. We write inverse as a^{-1}.

  Since commutative law applies in modulo, $a^{-1}a \equiv 1 \pmod n$.
- **Existence of multiplicative inverse** (4.7.3, NTP3):
  For any $a \in \mathbb{Z}$, its multiplicative inverse mod n (where $n > 1$),a^{-1}, exists iff, a, n are coprime
- **Corollary 4.7.4** (n is prime):
  If $n = p$ is prime, then all $a \in \mathbb{Z}, 0 < a < p$ have multiplicative inverses mod p
- **Cancellation Law** (8.4.9, Epp):
  $\forall a, b, c, n \in \mathbb{Z}, n > 1$ and a, n coprime, if $ab \equiv ac \pmod n$, then $b \equiv c \pmod n$

CS1231: Sequences & Recursion

Definitions

- **Sequences**
  Denote a seq. of numbers by: $a_0, a_1, a_2, \ldots$. $a_n = f(n)$, for some fn f and $n \in \mathbb{N}$. The indexing variable is n.

- **Recursion Relations**
  Seq. relating a_n to its predecessors: $a_{n-1}, a_{n-2}, \ldots$

Summation & Product

- **Summation**:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_{n-1} + a_n = S_n, \forall n \in \mathbb{N}$$

$$\sum_{i=m}^{n} a_i = \begin{cases} 0, & n < m \\ (\sum_{i=m}^{n-1} a_i) + a_n & \text{otherwise} \end{cases}$$

- **Product**:

$$\prod_{i=m}^{n} a_i = a_m \times a_{m+1} \times \ldots \times a_{n-1} \times a_n = P_n, \forall n \in \mathbb{N}$$

$$\prod_{i=m}^{n} a_i = \begin{cases} 1, & n < m \\ (\prod_{i=m}^{n-1} a_i) \cdot a_n & \text{otherwise} \end{cases}$$

- **Theorem 5.1.1** (Epp):
  If $a_m, a_{m+1}, \ldots$ and $b_m, b_{m+1}, \ldots$ are sequences of real numbers and for any $c \in \mathbb{R}$, then the following equations hold for any integer $n \geq m$:

  1. $\displaystyle\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$
  2. $\displaystyle c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$ (generalized distributive law)
  3. $\displaystyle\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k)$

Common Sequences

- **Arithmetic Sequence** ($a_n = a_{n-1} + d$)

$$\forall n \in \mathbb{N}, a_n = \begin{cases} a, & \text{if } n = 0, \\ a_{n-1} + d, & \text{otherwise.} \end{cases}$$

  Explicit Formula:

$$a_n = a + nd, \forall n \in \mathbb{N} \;\&\; a, r \in \mathbb{R}$$

  Closed Form:

$$S_n = \frac{n}{2}[2a + (n-1)d], \forall n \in \mathbb{N} \;\&\; a, r \in \mathbb{R}$$

- **Geometric Sequence** ($a_n = ra_{n-1}$)

$$\forall n \in \mathbb{N}, a_n = \begin{cases} a, & \text{if } n = 0, \\ ra_{n-1}, & \text{otherwise.} \end{cases}$$

  Explicit Formula:

$$a_n = ar^n, \forall n \in \mathbb{N} \;\&\; a, r \in \mathbb{R}$$

  Closed Form:

$$S_n = \frac{a(r_n - 1)}{r - 1}, \forall n \in \mathbb{N}, a, r \in \mathbb{R}$$

- **Square Numbers** (sum of first n odd numbers)
  Explicit Formula: $\forall n \in \mathbb{N}, \square_n = n^2$

- **Triangle Numbers** (sum of first $n+1$ integers)
  Explicit Formula: $\forall n \in \mathbb{N}, \triangle_n = \dfrac{n(n+1)}{2}$
  Interesting:

$$\forall n \in \mathbb{Z}^+, \triangle_n + \triangle_{n-1} = \square_n = (\triangle_n - \triangle_{n-1})^2$$

- **Fibonacci Numbers**

$$\forall n \in \mathbb{N}, F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$$

  Explicit Formula:

$$\forall n \in \mathbb{N}, F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}$$

  where $\phi = (1 + \sqrt{5})/2$

Solving Recurrences

- **Second-order Linear Homogeneous Recurrence Relation with Constant Coefficients** (Def 5.4.1, Slides)
  This is a recurrence relation in the form:

$$a_k = Aa_{k-1} + Ba_{k-2}, \forall k \in \mathbb{Z}_{\geq k_0}$$

  where $A, B \in \mathbb{R}$ constants, $B \neq$ and $k_0 \in \mathbb{Z}$ constant

- **Distinct-Roots Theorem** (Thm 5.8.3, Epp):
  Suppose a sequence $a_0, a_1, a_2, \ldots$ satisfies a recurrence relation

$$a_k = Aa_{k-1} + Ba_{k-2}$$

  for $A, B \in \mathbb{R}$ constants, with $B \neq 0$ and $k \in \mathbb{Z}_{\geq 2}$. If **characteristic equation**

$$t^2 - At - B = 0$$

  has two distinct roots r & s then $a_0, a_1, a_2, \ldots$ is given by **explicit formula**

$$a_n = Cr^n + Ds^n, \forall n \in \mathbb{N}$$

  where $C, D \in \mathbb{R}$ as determined by initial conditions a_0, a_1

- **Single-Roots Theorem** (Thm 5.8.5, Epp):
  Suppose a sequence $a_0, a_1, a_2, \ldots$ satisfies a recurrence relation

$$a_k = Aa_{k-1} + Ba_{k-2}$$

  for $A, B \in \mathbb{R}$ constants, with $B \neq 0$ and $k \in \mathbb{Z}_{\geq 2}$. If **characteristic equation**

$$t^2 - At - B = 0$$

  has a single real root r then $a_0, a_1, a_2, \ldots$ is given by **explicit formula**

$$a_n = Cr^n + Dnr^n, \forall n \in \mathbb{N}$$

  where $C, D \in \mathbb{R}$ as determined by the value a_0 and any other known value of the sequence

CS1231: Sets

Basics

- **Subset** (Def 6.1.1, Slides):
  S is **subset** of T (S is contained in T, T contains S) if all elements of S are elements of T. We write it as $S \subseteq T$

- **Empty set** (Def 6.3.1, Slides):
  Empty set has no element, denoted by $\varnothing$ or $\{\}$

- **Empty set is a subset of all sets** (6.2.4, Epp):
  $\forall X \forall Z((\forall Y {\sim}(Y \in X)) \rightarrow (X \subseteq Z))$

- **Set Equality** (Def 6.3.2, Slides):
  Two sets are equal iff they have same elements in them

  $$\forall X \forall Y((\forall Z(Z \in X \leftrightarrow Z \in Y)) \leftrightarrow X = Y)$$

  N.B. duplicates and order does not matter!

- **Prop 6.3.3**:
  For any sets X, Y; $X \subseteq Y$ & $Y \subseteq X$ iff, $X = Y$:

  $$\forall X \forall Y((X \subseteq Y \wedge Y \subseteq X) \leftrightarrow X = Y)$$

- **Empty Set is Unique** (Col 6.2.5, Epp):
  $\forall X_1 \forall X_2, ((\forall Y({\sim}(Y \in X_1)) \wedge (\forall Y {\sim}(Y \in X_2)) \rightarrow X_1 = X_2)$

- **Power Set** (Def 6.3.4, Slides):
  Given set S, the **power set of** S, denoted $\mathcal{P}(S)$ or 2^S, is the set whose elements are all the subsets of S, nothing less and nothing more.
  That is, given set S, if $T = \mathcal{P}(S)$, then

  $$\forall X((X \in T) \leftrightarrow (X \subseteq S))$$

- **No. of elements in Power Set** (Thm 6.3.1, Epp)
  For all integers $n \geq 0$, if a set X has n elements, then $\mathcal{P}(X)$ has 2^n elements.

Operation on Sets

- **Union** (Def 6.4.1):
  Let S be a set of sets, then we say that T is the **union** of the sets in S:

  $$T = \bigcup S = \bigcup_{X \in S} X$$

  iff each element of T belongs to some set S, nothing less and nothing more. That is, given S, T is such that:

  $$\forall Y((Y \in T) \leftrightarrow \exists Z((Z \in S) \wedge (Y \in Z)))$$

- **Proposition 6.4.2** (Slides):
  - $\bigcup \varnothing = \bigcup_{A \in \varnothing} A = \varnothing$
  - $\bigcup \{A\} = A$
  - $A \cup B = B \cup A$ (commutative)
  - $A \cup (B \cup C) = (A \cup B) \cup C$ (associative)
  - $A \cup A = A$
  - $A \subseteq B \leftrightarrow A \cup B = B$

- **Intersection** (Def 6.4.3, Slides):
  Let S be a **non-empty set** of sets. The **intersection** of the sets in S is the set T whose elements belong to all the sets in S, nothing less and more:

  $$\forall Y((Y \in T) \leftrightarrow \forall Z((Z \in S) \rightarrow (Y \in Z)))$$

  We write it as:

  $$T = \bigcap S = \bigcap_{X \in S} X$$

- **Proposition 6.4.4** (Slides):
  - $A \cap \varnothing = \varnothing$
  - $A \cap B = B \cap A$
  - $A \cap (B \cap C) = (A \cap B) \cap C$ (associative)
  - $A \subseteq B \leftrightarrow A \cap B = A$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- **Disjoint** (Def 6.4.5, Slides):
  S, T, being two sets, are disjoint iff $S \cap T = \varnothing$

- **Mutually disjoint** (Def 6.4.6, Slides):
  Let V be set of sets. The sets $T \in V$ are **mutually disjoint** iff every two distinct sets are disjoint.

  $$\forall X, Y \in V(X \neq Y \rightarrow X \cap Y = \varnothing)$$

  (e.g. $V = \{\{1,2\}, \{3\}, \{\{1\}, \{2\}\}\}$)

- **Partition** (Def 6.4.7, Slides):
  Let S be set and let V be a set of non-empty subsets of S. V is a **partition** of S iff

  1. The sets in V are mutually disjoint
  2. The union of the sets in V equals S.

- **Non-symmetric difference** (Def 6.4.8, Slides):
  Let S, T be two sets. The **difference** (or relative complement) of S and T, denoted $S - T$ is the set whose elements belong to S and do not belong to T

  $$\forall X(X \in S - T \iff (X \in S \wedge {\sim}(X \in T)))$$

- **Symmetric Difference** [XORing] (Def 6.4.9, Slides):
  Let S, T be two sets. The **symmetric difference** of S and T, denoted $S \ominus T$ is the set whose elements belong to S or T but not both

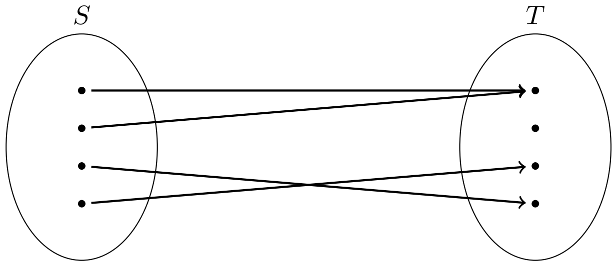
  $$\forall X(X \in S \ominus T \leftrightarrow (X \in S \oplus X \in T))$$

- **Set Complement** (Def 6.4.10, Slides):
  Let $\mathcal{U}$ be the Universal set, let $A \subseteq \mathcal{U}$. Then, the complement of A, denoted A^c, is $\mathcal{U} - A$s

CS1231: Functions

Basics

- **Function** (Def 7.1.1, Slides):



Let f be a relation such that $f \subseteq S \times T$. Then f is **function** from S to T ($f : S \to T$)

$$\forall x \in S, \exists! y \in T(x \ f \ y)$$

Basic Function Definitions

- **Pre-image** (Def 7.1.2):
Let $f : S \to T$ be a function. Let $x \in S$ and $y \in T$ such that $f(x) = y$. Then, x is the **pre-image** of y

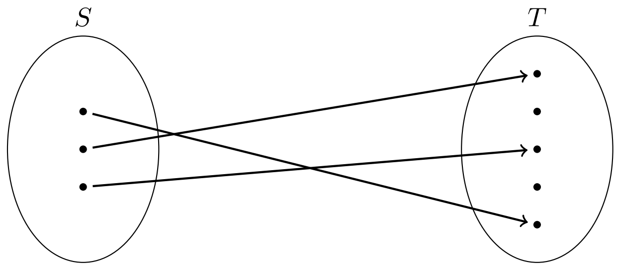
- **Inverse image** (Def 7.1.3):
Let $f : S \to T$ be a function. Let $y \in T$. The **inverse image** of y is the set of all its pre-images: $\{x \in S \mid f(x) = y\}$

- **Inverse image** (Def 7.1.4):
Let $f : S \to T$ be a function. Let $U \subseteq T$. The **inverse image** of U is the set that contains all the pre-images of all elements in U: $\{x \in S \mid \exists y \in U, f(x) = y\}$

- **Restriction** (Def 7.1.5):
Let $f : S \to T$ be a function. Let $U \subseteq S$. The **restriction** of f to U is the set: $\{(x, y) \in U \times T \mid f(x) = y\}$

Properties of Functions

- **Injective/One-to-One** (Def 7.2.1, Slides):

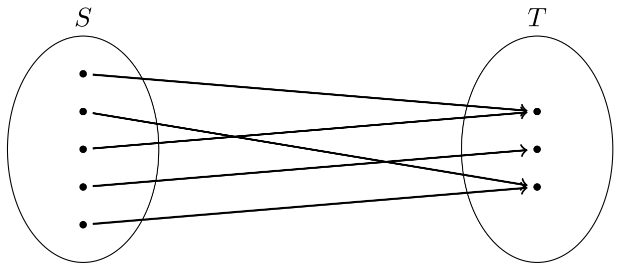


Let $f : S \to T$ be a function. f is **injective** iff

$$\forall Y \in T, \forall x_1, x_2 \in S((f(x_1) = y \land f(x_2) = y) \to x_1 = x_2)$$

We can also say: f is an **injection** or **one-to-one** (i.e. every dot in T has **AT MOST** one incoming arrow)

- **Surjective/Onto** (Def 7.2.2, Slides):

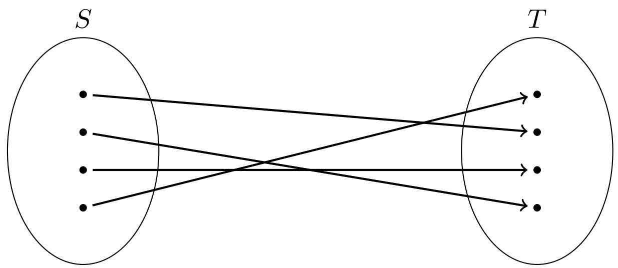


Let $f : S \to T$ be a function. f is **surjective** iff

$$\forall y \in T, \exists x \in S(f(x) = y)$$

We can also say: f is a **surjection** or **onto** (i.e. every dot in T has **AT LEAST** one incoming arrow)

- **Bijective** (Def 7.2.3, Slides):



Let $f : S \to T$ be a function. f is **bijective** iff f is both **injective and surjective**. We can also say: f is a **bijection**.

- **Inverse** (Prop 7.2.4, Slides):
Let $f : S \to T$ be a function and f^{-1} be the inverse relation o f from T to S. Then, f is bijective iff f^{-1} is a function.

- **Composition** (Prop 7.3.1, Slides):
Let $f : S \to T$ and $g : T \to U$ be functions. The composition of f and g, $g \circ f$, is a function from S to U.

- **Identity Function** (Def 7.3.2, Slides):
Given a set A, define function $\mathcal{I}_A$ from A to A by:

$$\forall x \in A(\mathcal{I}_A(x) = x)$$

$\mathcal{I}_A$ is the **identity function** on A

- **Composition of Inverse** (Prop 7.3.3, Slides):
Let $f : A \to A$ be injective function on A. Thus, $f^{-1} \circ f = \mathcal{I}_A$

Generalization

- **(n-ary) operation** (Def 7.3.4, Slides):
An **(n-ary) operation** on a set A is a function $f : \prod_1^n A \to A$. n is called the **arity** or **degree** of the operation.

- Unary operation (Def 7.3.5, Slides):
A **unary operation** on a set A is a function $f : A \to A$

- Binary operation (Def 7.3.6):
A **binary operation** on a set A is a function $f : A \times A \to A$

CS1231: Relations

Basic Definitions

- **Ordered Pair** (Def 8.1.1):
  Let S be a non-empty set and let $x, y \in S$. The **ordered pair**, denoted (x, y), is a mathematical object in which the first element is x and second element is y.

  $(x, y) = (a, b) \iff x = a, y = b$

- **Ordered n-tuple** (Def 8.1.2)

- **Cartesian product** (Def 8.1.3):
  Let S, T be two sets. The **Cartesian product** (cross product) of S & T, denoted $S \times T$, is the set such that:

  $$\forall X \forall Y ((X, Y) \in S \times T \leftrightarrow (X \in S) \wedge (Y \in T))$$

  N.B. Cartesian product is **NOT** commutative nor associative and size of $S \times T$ = size of $S \times$ size of T

- **Generalized Cartesian Product** (Def 8.1.4):
  If V is a set of sets, the Generalized Cartesian product of its elements is:

  $$\prod_{S \in V} S$$

- **Binary relations** (Def 8.2.1):
  Let S, T be two sets. A **binary relation** from S to T, denoted $\mathcal{R}$, is a subset of the Cartesian product $S \times T$

  N.B. $s \mathcal{R} t$ is $(s, t) \in \mathcal{R}$ and $s \not\mathcal{R} t$ is $(s, t) \notin \mathcal{R}$

Properties of Binary Relations

Let $R \subseteq S \times T$ be a binary relation from S to T

- **Domain** (Def 8.2.2):
  The **domain** of $\mathcal{R}$ is the set

  $$\mathcal{D}\text{om}(\mathcal{R}) = \{s \in S \mid \exists t \in T (s \mathcal{R} t)\}$$

- **Image** (Def 8.2.3):
  The **image** of $\mathcal{R}$ is the set

  $$\mathcal{I}\text{m}(\mathcal{R}) = \{t \in T \mid \exists s \in S (s \mathcal{R} t)\}$$

- **Co-domain** (Def 8.2.4):
  The **co-domain** (range) of $\mathcal{R}$ is the set

  $$\text{co}\mathcal{D}\text{om}(\mathcal{R}) = T$$

- **Inverse** (Def 8.2.6):
  Let S, T be sets and $R \subseteq S \times T$ be a binary relation. The **inverse** of the relation $\mathcal{R}$, denoted $\mathcal{R}^{-1}$, is the relation from T to S such that:

  $$\forall s \in S, \forall t \in T (t \mathcal{R}^{-1} s \leftrightarrow s \mathcal{R} t)$$

- **n-ary relation** (Def 8.2.7):
  Let S_i, for $i = 1$ to n, be n sets. An **n-ary relation** on the sets S_i, denoted $\mathcal{R}$, is a subset of the Cartesian product $\prod_{i=1}^{n} S_i$. We call n the **arity** or **degree** of the relation.

- **Composition** (Def 8.2.8):
  Let S, T, U be sets. Let $\mathcal{R} \subseteq S \times T$ be a relation. Let $\mathcal{R}' \subseteq T \times U$ be a relation. The composition of $\mathcal{R}$ with $\mathcal{R}'$, denoted $\mathcal{R} \circ \mathcal{R}'$, is the relation from S to U such that:

  $$\forall X \in S, \forall z \in U (x \mathcal{R}' \circ \mathcal{R} z \leftrightarrow (\exists y \in T (x \mathcal{R} y \wedge y \mathcal{R}' z)))$$

- **Associativity of Composition** (Prop 8.2.9):
  Let S, T, U, V be sets. Let $\mathcal{R} \subseteq S \times T$, $\mathcal{R}' \subseteq T \times U$, $\mathcal{R}'' \subseteq U \times V$ be relations. Therefore,

  $$\mathcal{R}'' \circ (\mathcal{R}' \circ \mathcal{R}) = (\mathcal{R}'' \circ \mathcal{R}') \circ \mathcal{R} = \mathcal{R}'' \circ \mathcal{R}' \circ \mathcal{R}$$

- **Proposition 8.2.10**:
  Let S, T, U be sets. Let $\mathcal{R} \subseteq S \times T$ and $\mathcal{R}' \subseteq T \times U$ be relations.

  $$(\mathcal{R}' \circ \mathcal{R})^{-1} = \underbrace{\mathcal{R}^{-1} \circ \mathcal{R}'^{-1}}_{\text{reversed order}}$$

Properties of Relations on a Set

Let A be a set and $\mathcal{R} \subseteq A \times A$ be a relation. We say that $\mathcal{R}$ is a **relation on** A.

- **Reflexive** (Def 8.3.1)
  $\mathcal{R}$ is **reflexive** $\iff \forall x \in A, (x \mathcal{R} x)$

- **Symmetric** (Def 8.3.2)
  $\mathcal{R}$ is **symmetric** $\iff \forall x, y \in A, (x \mathcal{R} y \rightarrow y \mathcal{R} x)$

- **Anti-Symmetric** (Def 8.6.1)
  $\mathcal{R}$ is **anti-symmetric** $\iff \forall x, y \in A, ((x \mathcal{R} y \wedge y \mathcal{R} x) \rightarrow x = y)$

- **Asymmetric** (Tutorial 7)
  $\mathcal{R}$ is **asymmetric** $\iff \forall x, y \in A, (x \mathcal{R} y \rightarrow y \not\mathcal{R} x)$

- **Transitive** (Def 8.3.3)
  $\mathcal{R}$ is **transitive** $\iff \forall x, y, z \in A, ((x \mathcal{R} y \wedge y \mathcal{R} z) \rightarrow x \mathcal{R} z)$

- **Equivalence Relations** (Def 8.3.4):
  Let $\mathcal{R}$ be a relation on set A.
  $\mathcal{R}$ is called an **equivalence relation** iff $\mathcal{R}$ is reflexive, symmetric and transitive.

- **Equivalence Class** (Def 8.3.5):
  Let $x \in A$ and $\mathcal{R}$ be an equivalence relation on A. The **equivalence class** of x, denoted $[x]$ is the set of all elemetns $y \in A$ that are in relation with x.

  $$[x] = \{y \in A \mid x \mathcal{R} y\}$$

- **Partition induced by an equivalence relation** (Thm 8.3.4, Epp):
  Let $\mathcal{R}$ be an equivalence relation on a set A. Then, the set of distinct equivalence classes form a partition of A.

- Lemma 8.3.2, Epp:
  Let $\mathcal{R}$ be an equivalence relation on a set A and let a, b be two elements in A. If $a \mathcal{R} b$ then $[a] = [b]$

- Lemma 8.3.3, Epp:
  If $\mathcal{R}$ is an equivalence relation on a set A and a, b are elements in A, then, either $[a] \cap [b] = \varnothing$ or $[a] = [b]$.

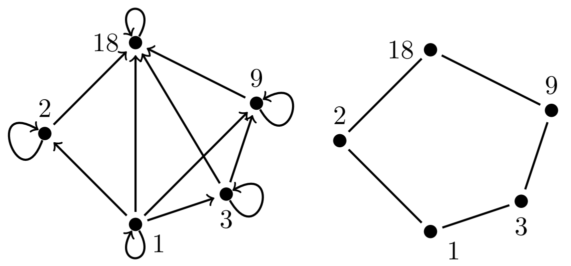
- **Equivalence relation induced by a partition** (Thm 8.3.1, Epp):
  Given a partition $S_1, S_2, \ldots$ of a set A, there exists an equivalence relation $\mathcal{R}$ on A whose equivalence classes make up precisely that partition.

Additional Definitions

- **Transitive closure** (Def 8.5.1)
  Let A be a set, $\mathcal{R}$ be a relation on A. The **transitive closure** of $\mathcal{R}$, denoted $\mathcal{R}^t$ is a relation that satisfies these three properties:
  1. $\mathcal{R}^t$ is transitive
  2. $\mathcal{R} \subseteq \mathcal{R}^t$
  3. If $\mathcal{S}$ is any other transitive relation such that $\mathcal{R} \subseteq \mathcal{S}$, then $\mathcal{R}^t \subseteq \mathcal{S}$
- **Repeated compositions**
  Let $\mathcal{R}$ be a relation on a set A. We adopt the following notation for the composition of $\mathcal{R}$ with itself:
  1. We define $\mathcal{R}^1 \triangleq \mathcal{R}$
  2. We define $\mathcal{R}^2 \triangleq \mathcal{R} \circ \mathcal{R}$
  3. We define $\mathcal{R}^n \triangleq \underbrace{\mathcal{R} \circ \ldots \circ \mathcal{R}}_{n} = \bigodot_{i=1 \text{ to } n} \mathcal{R}$
- **Proposition 8.5.2**
  Let $\mathcal{R}$ be a relation on set A. Then
  $$\mathcal{R}^t = \bigcup_{i=1}^{\infty} \mathcal{R}^i$$

Partial & Total Orders

- **Partial Order** (Def 8.6.2)
  $\mathcal{R}$ is said to be a **partial order** iff it is reflexive, anti-symmetric and transitive
  N.B. Partial order is denoted by $\preceq$ (note the curl)
- **Hasse Diagrams**
  To convert from left diagram to right diagram:



  N.B. Only works for partially ordered sets!

Converting to Hasse:

1. Draw the directed graph so that all arrows point upwards
2. Eliminate all self-loops
3. Eliminate all arrows implied by the transitive property
4. Remove the direction of the arrows

- **Comparable** (Def 8.6.3)
  Let $\preceq$ be a partial order on a set A. Elements $a, b \in A$ are **comparable** iff either $a \preceq b$ or $b \preceq a$. Otherwise, a, b are **noncomparable**.
- **Total Order** (Def 8.6.4)
  Let $\preceq$ be a partial order on a set A. $\preceq$ is a **total order** iff
  $$\forall x, y \in A(x \preceq y \vee y \preceq x)$$
  i.e. $\preceq$ is a total order if $\preceq$ is a partial order and all x, y are comparable
- **Maximal** (Def 8.6.5)
  An element x is a **maximal element** iff
  $$\forall y \in A, (x \preceq y \rightarrow x = y)$$
- **Maximum** (Def 8.6.6)
  An element, usually noted $\top$, is the **maximum element** iff
  $$\forall x \in A, (x \preceq \top)$$
- **Minimal** (Def 8.6.7)
  An element x is a **minimal element** iff
  $$\forall y \in A, (y \preceq x \rightarrow x = y)$$
- **Minimum** (Def 8.6.8)
  An element, usually noted $\perp$, is the **minimum element** iff
  $$\forall x \in A, (\perp \preceq x)$$
- **Well Ordering of Total Orders** (Def 8.6.9)
  Let $\preceq$ be a total order on a set A. A is **well ordered** iff every non-empty subset of A contains a minimum element
  $$\forall S \in \mathcal{P}(A)(S \neq \varnothing \rightarrow (\exists x \in S, \forall y \in S, (x \preceq y)))$$

CS1231: Counting & Probability

Basic Definition

- **Sample Space & Event**
  A **sample space** is the set of all possible outcomes of a random process or experiment. An **event** is a subset of a sample space.

- **Equally Likely Probability Formula**
  If S is a finite sample space in which all outcomes are **equally likely** & E is an event in S, then the **probability** of E, denoted $P(E)$ is

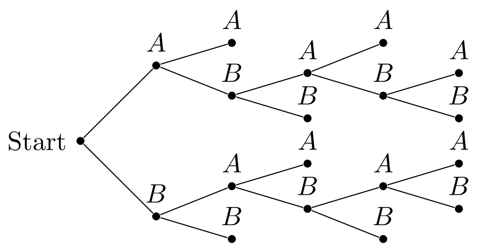
$$P(E) = \frac{\text{No. of outcomes in E}}{\text{Total no. of outcomes in S}} = \frac{N(E)}{N(S)}$$

- **Probability of the Complement of an Event**
  If S is a finite sample space and A is an event in S, then $P(A^c) = 1 - P(A)$

- **Number of Elements in a List** (Thm 9.1.1)
  If $m, n \in \mathbb{Z}$ and $m \leq n$, then there are $\boldsymbol{(n-m)+1}$ integers from m to n inclusive

- **Multiplication Rule** (Thm 9.2.1)
  If an operation consists of k steps and
  the first step can be preformed in n_1 ways
  the second step can be performed in n_2 ways
  (regardless of how first step was performed)
  $\vdots$
  the kth step can be performed in n_k ways
  (regardless of how preceding steps was performed)
  Then, the entire operation can be performed in $n_1 \times n_2 \times \ldots \times n_k$ ways

Possibility Tree



- **Possible Ways in Tree**
  Possible ways are represented by the distinct paths from "root" (start) to "leaf" (terminal point) in the tree

Permutation

- **Definition**
  A **permutation** of a set of objects is an ordering of the objects in a row.

- **No of Permutations** (Thm 9.2.2)
  The **number of permutations** of a set with n ($n \geq 1$) elements is $n!$

- **r-permutation**
  An **r-permutation** of a set of n elements is an ordered selection of r elements taken from the set. The number of r-permutations of a set of n elements is denoted $P(n, r)$

- **r-permutations from a set of n elements** (Thm 9.2.3)
  If $n, r \in \mathbb{Z}$ and $1 \leq r \leq n$, then the **number of r-permutations of a set of n elements** is given by the formula

$$P(n, r) = n(n-1)\ldots(n-r+1)$$

or, equivalently

$$P(n, r) = \frac{n!}{(n-r)!}$$

Counting Elements of Sets

- **Addition Rule** (Thm 9.3.1)
  Suppose a finite set A equals the union of k distinct mutually disjoint subsets $A_1, A_2, \ldots, A_k$. Then, $N(A) = N(A_1) + N(A_2) + \ldots + N(A_k)$

- **Difference Rule** (Thm 9.3.2)
  If A is a finite set and B is a subset of A, then $N(A - B) = N(A) - N(B)$

- **Inclusion-Exclusion rule for 2 or 3 sets** (Thm 9.3.3)
  If A, B, C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

and

$$\begin{aligned} N(A \cup B \cup C) = {} & N(A) + N(B) + N(C) - N(A \cap B) \\ & - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C) \end{aligned}$$

Pigeonhole Principle

- **Pigeonhole Principle**
  A function from one finite set to a smaller finite set cannot be one-to-one: There must be at least 2 elements in the domain that have the same image in the co-domain

- **Pigeonhole Principle** (Thm 9.4.1)
  For any function f from a finite set X with n elements to a finite set Y with m elements, if $n > m$, then f is not one-to-one.

- **One-to-one and Onto for Finite Sets**
  Let X, Y be finite sets with the same number of elements and suppose f is a function from X to Y. Then f is one-to-one iff f is onto.

- **Generalised Pigeonhole Principle**
  For any function f from a finite set X with n elements to a finite set Y with m elements and for any $k \in \mathbb{Z}^+$, if $k < n/m$, then there is some $y\ inY$ such that y is the image of at least $k+1$ distinct elements of X.

- **Contrapositive Form of GPP**
  For any function f from a finite set X with n elements to a finite set Y with m elements and for any $k \in \mathbb{Z}^+$, if for each $y \in Y$, $f^{-1}(y)$ has at most k elements, then X has at most km elements, i.e., $n \leq km$

Combinations

- **r-combination**
Let n, r be non-negative integers with $r \leq n$. An **r-combination** of a set of n elements is a subset of r of the n elements. $\binom{n}{r}$, denotes the no. of subsets of size r, that can be chosen from a set of n elements.

- **Formula for $\binom{n}{r}$** (Thm 9.5.1)
The no. of subsets of size r (r-combinations) that can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula
$$\binom{n}{r} = \frac{P(n,r)}{r!}$$
or, equivalently
$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$
where n, r are non-negative integers with $r \leq n$.

- **Permutations with sets of indistinguishable objects** (Thm 9.5.2)
Suppose a collection consists of n objects of which

  n_1 of type 1 & are indistinguishable from each other
  n_2 of type 2 & are indistinguishable from each other
  $\vdots$
  n_k of type k & are indistinguishable from each other

  and suppose that $n_1 + n_2 + \ldots + n_k = n$. Then, the **no. of distinguishable permutations** of the n objects is
$$\binom{n}{n_1}\binom{n-n_1}{n_2}\ldots\binom{n-n_1-n_2-\ldots n_{k-1}}{n_k} = \frac{n!}{n_1!n_2!\ldots n_k!}$$

- **No. of Partitions of a Set into r subsets**
(Stirling numbers of the Second Kind)

  $S_{n,r}$ = no. of ways a set of size n can be partitioned into r subsets

- **r-combination with repetition**
An **r-combination with repetition allowed**, or **multiset of size r**, chosen from a set X of n elements is an unordered selection of elements taken from X with repetition allowed. If $X = \{x_1, x_2, \ldots, x_n\}$, we write an r-combination with repetition allowed as $[x_{i_1}, x_{i_2}, \ldots, x_{i_r}]$ where each x_{i_j} is in X and some of the x_{i_j} may equal each other

- **No. of r-combinations with repetition** (Thm 9.6.1)
The **no. of r-combination with repetition allowed** (multisets of size r) that can be selected from a set of n elements is
$$\binom{r+n-1}{r}$$
This equals the number of ways r objects can be selected from n categories of objects with repetitions allowed

- **Pascal's Formula** (Thm 9.7.1)
Suppose $n, r \in \mathbb{Z}^+$ & $r \leq n$. Then
$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

- **Binomial Theorem** (Thm 9.7.2)
Given any $a, b \in \mathbb{R}$ and any non-negative integer n,
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$
$$= a^n + \binom{n}{1} a^{n-1} b^1 + \ldots + \binom{n}{n-1} a^1 b^{n-1} + b^n$$

Probability

- **Probability Axioms**
Let S be a sample space. A **probability function** P from the set of all events in S to the set of real numbers satisfies the following axioms: For all events A and B in S,

  1. $0 \leq P(A) \leq 1$
  2. $P(\varnothing) = 0$ and $P(S) = 1$
  3. If A and B are disjoint ($A \cap B = \varnothing$), then $P(A \cup B) = P(A) + P(B)$

- **Probability of a General Union of Two Events**
If A and B are any events in a sample space S, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

- **Expected Value**
Suppose the possible outcomes of an experiment, or random process, are real numbers $a_1, a_2, \ldots, a_n$ which occur with probabilities $p_1, p_2, \ldots, p_n$. The **expected value** of the process is
$$\sum_{k=1}^{n} a_k p_k = a_1 p_1 + a_2 p_2 + \ldots + a_n p_n$$

- **Conditional Probability**
Let A and B be events in a sample space S. If $P(A) \neq 0$, then the **conditional probability of B given A**, denoted $P(B \mid A)$ is
$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

- **Bayes' Theorem** (Thm 9.9.1)
Suppose that a sample space S is a union of mutually disjoint events $B_1, B_2, \ldots, B_n$.
Suppose A is an event in S, and suppose A and all the B_i have non-zero probabilities.
If $k \in \mathbb{Z}$ with $1 \leq k \leq n$, then
$$P(B_k \mid A) = \frac{P(A \mid B_k) \cdot P(B_k)}{P(A \mid B_1) \cdot P(B_1) + \ldots + P(A \mid B_n) \cdot P(B_n)}$$

- **Independent Events**
If A, B are events in a sample space S, then A and B are **independent** iff $P(A \cap B) = P(A) \cdot P(B)$

- **Pairwise/Mutually Independent**
Let A, B, C be events in a sample space S. A, B, C are **pairwise independent**, iff, they satisfy conditions 1 - 3 below. They are **mutually independent**, if, they satisfy all four conditions below.

  1. $P(A \cap B) = P(A) \cdot P(B)$
  2. $P(A \cap C) = P(A) \cdot P(C)$
  3. $P(B \cap C) = P(B) \cdot P(C)$
  4. $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

- **Generalised Mutually Independent Definition**
Events $A_1, A_2, \ldots, A_n$ in a sample space S are **mutually dependent** iff the probability of the intersection of any subset of the events is the product of the probabilities of the events in the subset

CS1231: Graphs

Basic Definitions

- **Graph**
  A **graph** G consists of 2 finite sets: a nonempty set $V(G)$ of **vertices** and a set $E(G)$ of **edges**, where each edge is associated with a set consisting of either one or two vertices called its **endpoints**.

  A edge is said to **connect** its endpoints; two vertices that are connected by an edge are called **adjacent vertices**; and a vertex that is an endpoint of a loop is said to be **adjacent to itself**.

  An edge is said to be **incident on** each of its endpoints, and two edges incident on the same endpoint are called **adjacent edges**.

  We write $e = \{v, w\}$ for an edge e incident on vertices v and w.

- **Directed Graph**
  A **directed graph**, or **digraph**, G, consists of 2 finite sets: a nonempty set V(G) of **vertices** and a set $D(G)$ of **directed edges**, where each edge is associated with an ordered pair of vertices called its endpoints.

  If edge e is associated with the pair (v, w) of vertices, then e is said to be the **(directed) edge** from v to w. We write $e = (v, w)$.

- **Simple Graph**
  A **simple graph** is a undirected graph that does **not** have any loops or parallel edges.

- **Complement of Simple Graph** (Tutorial 10)
  If G is a simple graph, the complement of G, denoted G', is obtained as follows: The vertex set of G' is identical to the vertex set of G. However, two distinct vertices v and w of G' are connected by an edge iff v & w are not connected by an edge in G.

- **Complete Graph**
  A **complete graph** on **n vertices**, $n > 0$, denoted $\boldsymbol{K_n}$, is a simple graph with n vertices and exactly one edge connecting each pair of distinct vertices.

- **Subgraph of a Graph**
  A graph H is said to be a **subgraph** of graph G, iff, every vertex in H is also a vertex in G, every edge in H is also an edge in G, and every edge in H has the same endpoints as it has in G.

- **Complete Bipartite Graphs**
  A **complete bipartite graph** on $\boldsymbol{(m, n)}$ **vertices**, where $m, n > 0$, denoted $\boldsymbol{K_{m,n}}$, is a simple graph with distinct vertices $v_1, v_2, \ldots, v_m$ and $w_1, w_2, \ldots, w_n$ that satisfies the following properties:

  For all $i, k = 1, 2, \ldots, m$ and for all $j, l = 1, 2, \ldots, n$,

  1. There is an edge from each vertex v_i to each vertex w_j
  2. There is no edge from any vertex v_i to any other vertex v_k
  3. There is no edge from any vertex w_j to any other vertex w_l

- **Degree of a Vertex and Total Degree of a Graph**
  Let G be a graph and v a vertex of G. The **degree** of v, denoted $\boldsymbol{deg(v)}$, equals the number of edges that are **incident on** $\boldsymbol{v}$, with an edge that is a loop counted twice.
  The **total degree of** $\boldsymbol{G}$ is the sum of the degrees of all the vertices of G.

- **Handshake Theorem** (Thm 10.1.1)
  If G is any graph, then the sum of the degrees of all the vertices of G equals twice the number of edges of G. Specifically, if the vertices of G are $v_1, v_2, \ldots, v_n$, where $n \geq 0$, then

$$\begin{aligned}\text{Total degree of } G &= \deg(v_1) + \deg(v_2) + \ldots + \deg(v_n) \\ &= 2 \cdot (\text{the no. of } \textbf{edges} \text{ of } G)\end{aligned}$$

- **Corollary 10.1.2**
  The total degree of a graph is **even**

- **Proposition 10.1.3**
  In any graph there are an even numer of vertices of odd degree

Trails, Paths and Circuits

Let G be a graph and let v, w be vertices of G.

- **Walk**
  A **walk from** $\boldsymbol{v}$ **to** $\boldsymbol{w}$ is a finite alternating sequence of adjacent vertices and edges of G. Thus, a walk has the form

$$v_0 e_1 v_1 e_2 \ldots v_{n-1} e_n v_n$$

  where the v's represent vertices, the e's represent edges, $v_0 = v, v_n = w$ and for all $i \in \{1, 2, \ldots, n\}, v_{i-1}$ and v_i are the endpoints of e_i

- **Trivial Walk**
  A **trivial walk** from v to v consists of the single vertex v

- **Trail**
  A **trail from** $\boldsymbol{v}$ **to** $\boldsymbol{w}$ is a walk from v to w that does not contain a repeated edge

- **Path**
  A **path from** $\boldsymbol{v}$ **to** $\boldsymbol{w}$ is a trail that does not contain a repeated vertex

- **Closed Walk**
  A **closed walk** is a walk that starts and ends at the same vertex

- **Circuit/Cycle**
  A **circuit/cycle** is a **closed walk** thtat contains at least one edge and does not contain a repeated edge

- **Simple Circuit/Cycle**
  A **simple circuit/cycle** is a circuit that does not have any other repeated vertex except first and last

- **Triangle**
  A simple circuit of **length three** is called a triangle

- **Connectedness**
  Vertices v and w in graph G are **connected** iff there is a walk from v to w

- **Connected Graph**
  The graph G is **connected**, iff, given any two vertices v and w in graph G, there is a walk from v to w

- **Lemma on Connectedness**
  Let G be a graph
  1. If G is connected, then any two distinct vertices in G can be connected by a path
  2. If vertices v and w are part of a circuit in G and one edge is removed from the circuit, then there still exists a trail from v to w in G
  3. If G is connected and G contains a circuit, then an edge of the circuit can be removed without disconnected G

- **Connected Component**
  A graph H is a **connected component** of a graph G iff,
  1. The graph H is a subgraph of G
  2. The graph H is connected
  3. No connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H

Euler Circuits

- **Euler Circuit**
  Let G be a graph. An **Euler circuit** for G is a circuit that contains every vertex and every edge of G.

  That is, an **Euler circuit** for G is a sequence of adjacent vertices and edges in G that has at least one edge, starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.

- **Theorem 10.2.2**
  If a graph is an Euler circuit, then every vertex of the graph has positive even degree

- **Contrapositive of 10.2.2**
  If some vertex of a graph has odd degree, then the graph does not have an Euler circuit

- **Theorem 10.2.3**
  If a graph G is **connected** and the degree of every vertex of G is a **positive even integer**, then G has an Euler circuit

- **Theorem 10.2.4: USE THIS FOR EULER**
  A graph G has an Euler circuit $\iff$ G is connected and every vertex of G has positive even degree

- **Euler Trail**
  Let G be a graph, and let v and w be two distinct vertices of G. An **Euler trail/path from v to w** is a sequence of adjacent edges and vertices that starts at v, ends at w, passes through every vertex of G at least once, and traverses every edge of G exactly once.

- **Corollary 10.2.5**
  Let G be a graph and let v and w be two distinct vertices of G. There is an Euler trail from v to w $\iff$ G is connected, v and w have odd degree, and all other vertices of G have positive even degree.

Hamiltonian Circuits

- **Hamiltonian Circuit**
  Given a graph G, a **Hamiltonian circuit** for G is a simple circuit that includes every vertex of G.

  That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once, **except for the first and last**, which are the same.

  N.B. Hamiltonian circuit does not have to use all edges, but since it is a circuit, it cannot use the same edge more than once.

- **Proposition 10.2.6**
  If a graph G has a Hamiltonian circuit, then G has a subgraph H with the following properties:
  1. H contains every vertex of G
  2. H is connected
  3. H has the same number of edges as vertices
  4. Every vertex of H has degree 2

Matrix Representation of Graphs

- **Matrix**
  An $m \times n$ **matrix** A over a set S is a rectangular array of elements of S arranged into m rows and n columns

- **Adjacency Matrix of a Directed Graph**
  Let G be a directed graph with ordered vertices $v_1, v_2, \ldots, v_n$. The **adjacency matrix of G** is the $n \times n$ matrix $\mathbf{A} = (a_{ij})$ over the set of non-negative integers such that

  $$a_{ij} = \text{the number of arrows from } v_i \text{ to } v_j$$

  for all $i, j = 1, 2, \ldots, n$.

- **Adjacency Matrix of an Undirected Graph**
  let G be an undirected graph with ordered verices $v_1, v_2, \ldots, v_n$. The **adjacency matrix of G** is the $n \times n$ matrix $\mathbf{A} = (a_{ij})$ over the set of non-negative integers such that

  $$a_{ij} = \text{the number of edges connecting } v_i \text{ and } v_j$$

  for all $i, j = 1, 2, \ldots, n$.

- **Symmetric Matrix**
  An $n \times n$ square matrix $\mathbf{A} = (a_{ij})$ is called **symmetric** $\iff$ for all $i, j = 1, 2, \ldots, n$

  $$a_{ij} = a_{ji}$$

  (i.e. mirror image along main diagonal)

- **Theorem 10.3.1**
  Let G be a graph with connected componenets $G_1, G_2, \ldots, G_k$. If there are n_i vertices in each connected component G_i and these vertices are numbered consecutively, then the adjacency matrix of G has the form:

  $$\begin{bmatrix} A_1 & O & O & & O & O \\ O & A_2 & O & \ldots & O & O \\ O & O & A_3 & & O & O \\ & \vdots & & & \vdots & \vdots \\ O & O & O & \ldots & O & A_k \end{bmatrix}$$

  where each A_i is $n_i \times n_i$ adjacency matrix of G_i for all $i = 1, 2, \ldots, k$, and the O's represent matrices whose entries are all 0s

- **Scalar Product**
  Suppose that all entries in matrices **A** and **B** are real numbers. If the number of elements, n, in the ith row of **A** equals the number of elements in the jth column of **B**, then the **scalar product** or **dot product** of the ith row of **A** and the jth column of **B** is the real number obtained as follows

$$\begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{in}b_{nj}$$

- **Matrix Multiplication**
  Let $\mathbf{A} = (a_{ij})$ be an $m \times k$ matrix and $\mathbf{B} = (b_{ij})$ an $k \times n$ matrix with real entries. The (matrix) product of **A** times **B**, denoted **AB**, is the matrix (c_{ij}) defined as follows:

$$\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1j} & \vdots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2j} & \vdots & c_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \dots & c_{ij} & \vdots & c_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{k1} & c_{k2} & \dots & c_{1j} & \vdots & c_{kn} \end{bmatrix}$$

  where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ik}bkj = \sum_{r=1}^{k} a_{ir}b_{rj}$$

  for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

- **Identity Matrix**
  For each $n \in \mathbb{Z}^+$, the $\boldsymbol{n \times n}$ **identity matrix**, denoted $I_n = (\delta_{ij})$ or just **I**, if the size of matrix is obvious from context, is the $n \times n$ matrix in which all the entries in the main diagonal are 1's and all other entries are 0's. In other words,

$$\delta_{ij} = \begin{cases} 1, \text{if } i = j \\ 0, \text{if } i \neq j \end{cases}$$

  for all $i, j = 1, 2, \ldots, n$.

- **Identity Matrix II**
  For any $n \times n$ matrix **A**, the **powers of A** are defined as follows:
  $\mathbf{A^0} = \mathbf{I}$ where **I** is the $n \times n$ identity matrix
  $\mathbf{A^n} = \mathbf{AA^{n-1}}$ for all integers $n \geq 1$

- **No. of walks in Adjacency Matrix** (Thm 10.3.2)
  If G is a graph with vertices $v_1, v_2, \ldots, v_m$ and **A** is the adjacency matrix of G, then for each $n \in \mathbb{Z}^+$ and for all integers $i, j = 1, 2, \ldots, m$, the ij-th entry of $\mathbf{A}^n =$ the number of walks of length n from v_i to v_j.

Isomorphisms of Graphs

- **Isomorphic Graph**
  Let G and G' be graphs with vertex sets $V(G)$ and $V(G')$ and edge sets $E(G)$ and $E(G')$ respectively. $\boldsymbol{G}$ **is isomorphic to** $\boldsymbol{G'}$ $\iff$ there exist one-to-one correspondences $g : V(G) \to V(G')$ and $h : E(G) \to E(G')$ that preserve the edge-endpoint functions of G and G' in the sense that for all $v \in V(G)$ and $e \in E(G)$,

  v is an endpoint of e $\iff$ $g(v)$ is an endpoint of $h(e)$.

- **Graph Isomorphism is an Equivalence Relation** (Thm 10.4.1)
  Let S be a set of graphs and let R be the relation of graph isomorphism on S. Then, R is an equivalence relation on S.

- **Self-Complementary Graph** (Tutorial 10)
  A **self-complementary graph** is isomorphic with its complement

- **Invariant-ness**
  A property P is called an **invariant** for graph isomorphism $\iff$ given any graphs G ad G', if G has property P and G' is isomorphic to G, then G' has property P.

- **Invariants for Graph Isomorphism** (Thm 10.4.2)
  Each of the following properties is an invariant for graph isomorphism, where n, m, k are all non-negative integers

  1. has n vertices
  2. has m edges
  3. has a vertex of degree k
  4. has m vertices of degree k
  5. has a circuit of length k
  6. has a simple circuit of length k
  7. has m simple circuits of length k
  8. is connected
  9. has an Euler circuit
  10. has a Hamiltonian circuit

- **Graph Isomorphism for Simple Graphs**
  If G and G' are simple graphs, then $\boldsymbol{G}$ **is isomorphic to** $\boldsymbol{G'}$ $\iff$ there exists a one-to-one correspondence g from vertex set $V(G')$ of G' that preserves the edge-endpoint functions of G and G' in the sense that for all vertices u and v of G,

  $\{u, v\}$ is an edge in G $\iff$ $\{g(u), g(v)\}$ is an edge in G'

CS1231: Trees

Basic Definition

- **Circuit-Free**
  A **graph** is said to be **circuit-free** $\iff$ it has no circuits
- **Tree**
  A graph is said to be a **tree** $\iff$ it is circuit-free and connected
- **Trivial Tree**
  A **trivial tree** is a graph that consists of a single vertex
- **Forest**
  A graph is called a **forest** $\iff$ it is circuit-free and not connected
- **Minimum vertex of non-trivial tree** (Lem 10.5.1)
  Any non-trivial tree has at least one vertex of degree 1
- **Terminal vertex (leaf) & internal vertex**
  Let T be a tree. If T has only one or two vertices, then each is called a **terminal vertex** (or **leaf**). If T has at least three vertices, then a vertex of degree 1 in T is called a **terminal vertex** (or **leaf**), and a vertex of degree greater than 1 in T is called an **internal vertex**.
- **Theorem 10.5.2**
  Any tree with n vertices ($n > 0$) has $n - 1$ edges
- **Lemma 10.5.3**
  If G is any connected graph, C is any circuit in G, and one of the edges of C is removed from G, then the graph that remains is still connected
- **Determining a Tree** (Thm 10.5.4)
  If G is a connected graph with n vertices and $n - 1$ edges, then G is a tree

Rooted Trees

- **Rooted Tree**
  A **rooted tree** is a tree in which there is one vertex that is distinguished from the others and is called the **root**
- **Level**
  The **level** of a vertex is the number of edges along the unique path between it and the root
- **Height**
  The **height** of a rooted tree is the maximum level of any vertex of the tree
- **Children**
  Given the root or any internal vertex v of a rooted tree, the **children** of v are all those vertices that are adjacent to v and are one level farther away from the root than v
- **Parent**
  If w is a child of v, then v is called the **parent** of w, and two distinct vertices that are both children of the same parent are called **sibilings**
- **Ancestor/Descendant**
  Given two distinct vertices v and w, if v lies on the unique path between w and the root, then v is an **ancestor** of w, and w is a **descendant** of v

Binary Trees

- **Binary Tree**
  A **binary tree** is a rooted tree in which every parent has **at most two children**. Each child is designated either a **left child** or a **right child** (but not both), and every parent has at most one left and one right child.
- **Full Binary Tree**
  A **full binary tree** is a binary tree in which each parent has **exactly two children**
- **Left/Right Subtree**
  Given any parent v in a binary tree T, if v has a left child, then the **left subtree** of v is the binary tree whose root is the left child of v, whose vertices consist of the left child of v and all its descendants, and whose edges consists of all those edges of T that connect the vertices of the left subtree

  The **right subtree** of v is defined analogously
- **Full Binary Tree Theorem** (Thm 10.6.1)
  If T is a full binary tree with k internal vertices, then T has a total of $2k + 1$ vertices and has $k + 1$ terminal vertices (leaves)
- **Maximum no. of terminal vertices** (Thm 10.6.2)
  For non-negative integers h, if T is any binary tree with height h and t terminal vertices (leaves), then

  $$t \leq 2^h$$

  Equivalently: $\log_2 t \leq h$

Binary Tree Traversal

- **Breath-First Search**
  In BFS, start at the root and visit the adjacent vertices, then move on to the next level
- **Depth-First Search**
  There are three kinds of DFS, **pre-order**, **in-order** and **post-order**.

  Pre-Order
  - *Print the data of the root (or current vertex)*
  - Traverse the **left** subtree by recursively calling the pre-order $f(x)$
  - Traverse the **right** subtree by recursively calling the pre-order $f(x)$

  In-Order
  - Traverse the **left** subtree by recursively calling the pre-order $f(x)$
  - *Print the data of the root (or current vertex)*
  - Traverse the **right** subtree by recursively calling the pre-order $f(x)$

  Post-Order
  - Traverse the **left** subtree by recursively calling the pre-order $f(x)$
  - Traverse the **right** subtree by recursively calling the pre-order $f(x)$
  - *Print the data of the root (or current vertex)*

Spanning Trees & Shortest Paths

- **Spanning Tree**
  A **spanning tree** for a graph G is a subgraph of G that contains every vertex of G and is a tree

- **Proposition 10.7.1**

  1. Every connected graph has a spanning tree
  2. Any two spanning trees for a graph have the same no. of edges

- **Weighted Graph**
  A **weighted graph** is a graph for which each edge has an associated positive real number **weight**. The sum of the weights of all edges is the **total weight** of the graph

- **Minimum Spanning Tree**
  A **minimum spanning tree** for a connected weighted graph is a spanning tree that has the least possible total weight compared to all other spanning trees for the graph

- $\boldsymbol{w(e)}$ **&** $\boldsymbol{w(G)}$
  If G is a weighted graph and e is an edge of G, then $\boldsymbol{w(e)}$ denotes the weight of e and $\boldsymbol{w(G)}$ denotes the total weight of G

- **Kruskal's Algorithm** (Alg 10.7.1)
  Input: G [a connected weighted graph with n vertices]
  Algorithm:

  1. Init T to have all the vertices of G and no edges
  2. Let E be the set of all edges of G, and let $m = 0$
  3. While $(m < n - 1)$
     (a) Find an edge e in E of least weight
     (b) Delete e from E
     (c) If addition of e to the edge set of T does not produce a circuit, then add e to the edge set of T and set $m = m + 1$
  4. End While

  Output: T [T is the MST for G]

- **Prim's Algorithm** (Alg 10.7.2)
  Input: G [a connected weighted graph with n vertices]
  Algorithm:

  1. Pick a vertex v of G and let T be the graph with this vertex only
  2. Let V be the set of all vertices of G except v.
  3. For $i = 1$ to $n - 1$
     (a) Find an edge e of G such that (1) e connects T to one of the vertices in V, and (2) e has the least weight of all edges connecting T to a vertex in V. Let w be the endpoint of e that is in V
     (b) Add e and w to the edge and vertex sets of T, delete w from V.

  Output: T [T is the MST for G]

- **Dijkstra's Algorithm** (Alg 10.7.3)
  Input: G [a connected weighted graph with positive weight for every edge], ∞ [a no. greater than the sum of the weights of all the edges in G], $w(u, v)$ [the weight of edge $\{u, v\}$], a [the source vertex], z [the dest vertex]
  Algorithm:

  1. init T to be the graph with vertex a and no edges. Let $V(T)$ be the set of vertices of T, and let $E(T)$ be the set of edges of T
  2. Let $L(a) = 0$, and for all vertices in G except a, let $L(u) = \infty$ [The number $L(x)$ is called the label of x]
  3. Init v to equal a and F to be $\{a\}$. [The symbol v is used to denote the vertex most recently added to T]
  4. Let $Adj(x)$ denote the set of vertices adjacent to vertex x
  5. while $(z \notin V(T))$
     (a) $F \leftarrow (F - \{v\}) \cup \{\text{vertices} \in \text{Adj}(v) \text{ and } \notin V(T)\}$ [Set F is set of fringe vertices]
     (b) For each vertex $u \in \text{Adj}(v)$ and $\notin V(T)$,
       – If $L(v) + w(v, u) < L(u)$ then
       – $L(u) \leftarrow L(v) + w(v, u)$
       – $D(u) \leftarrow v$
     (c) Find a vertex x in F with the smallest label. Add vertex x to $V(T)$, and add edge $\{D(x), x\}$ to $E(T)$. $v \leftarrow x$

  Output: $L(z)$ [this is the length of the shortest path from a to z]

