Tournaments in Computational Social Choice

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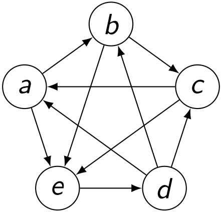
Outline

- **Social choice theory**
  - How to choose a socially desirable outcome from a set of alternatives.
  - Origins in mathematics, economics, and political science.
  - Tournaments model scenarios in which decisions are made based on pairwise comparisons.
  - Sports, elections, webpage ranking, biological interactions, ...
- **This tutorial**
  - Part 1: Tournament solutions (methods for choosing winners from a given tournament)
  - Part 2: Single-elimination tournaments (setting up the bracket to help a certain player, bribery issues)
  - Based partially on my survey published at IJCAI 2021.
  - For work before 2016, see Chapters 3 and 19 in the Handbook of Computational Social Choice [Brandt et al. '16]

W. Suksompong. "Tournaments in computational social choice: Recent developments", IJCAI 2021
F. Brandt, V. Conitzer, U. Endriss, J. Lang, A. D. Procaccia. "Handbook of Computational Social Choice", 2016

Part 1: Tournament Solutions

Tournaments



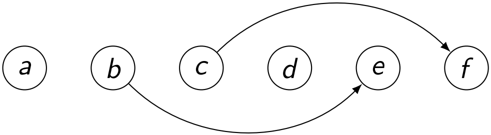
1: $a \succ_1 b \succ_1 c \succ_1 e \succ_1 d$
2: $d \succ_2 c \succ_2 a \succ_2 b \succ_2 e$
3: $e \succ_3 d \succ_3 b \succ_3 c \succ_3 a$

Tournament $T = (A, \succ)$, where A is the set of alternatives and $\succ$ is the dominance relation. In this example, $A = \{a, b, c, d, e\}$ and $a \succ b$, $b \succ c$, $d \succ b$, $e \succ d$, etc.

Tournament Solutions

- A tournament solution is a method for choosing the "winners" of any tournament.
- Copeland set (CO): Alternatives with the highest outdegree.
- Top cycle (TC): Alternatives that can reach every other alternative via a directed path (which by definition has length $\leq n - 1$).
- Uncovered set (UC): Alternatives that can reach every other alternative via a directed path of length ≤ 2.

Example

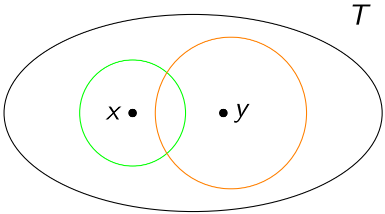


All omitted edges point from right to left.

- Outdegrees:
  - a: 0, b: 2, c: 3, d: 3, e: 3, f: 4
- $CO = \{f\}$
- $TC = \{b, c, d, e, f\}$
- $UC = \{c, d, e, f\}$
- No Condorcet winner (alternative that dominates all other alternatives), but a Condorcet loser (a)

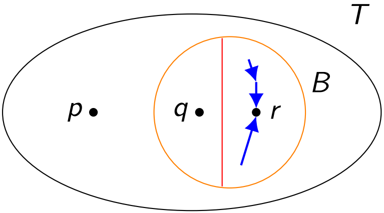
Top Cycle

- Equivalent definition of the top cycle:
  - (Unique) smallest nonempty set B of alternatives such that all alternatives in B dominate all alternatives outside B.



Top Cycle

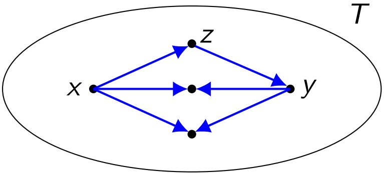
- Equivalent definition of TC:
  - (Unique) smallest nonempty set B of alternatives such that all alternatives in B dominate all alternatives outside B.



- Proof of equivalence:
  - $p \notin B$ cannot reach $q \in B$, so p does not belong to TC.
  - $q \in B$ can reach $p \notin B$ directly.
  - If $q \in B$ could not reach $r \in B$, all alternatives that could reach r would form a smaller subset in the definition of TC, contradiction.

Uncovered Set

- Covering relation: An alternative x covers another alternative y if
  - x dominates y.
  - For any z, if y dominates z, then x also dominates z.
- Strong indicator that x is better than y.
- Equivalent definition of UC:
  - The set of all uncovered alternatives.
  - **Proof:** x can reach y in ≤ 2 steps $\Longleftrightarrow$ y does not cover x.



- $CO \subseteq UC \subseteq TC$ always holds.

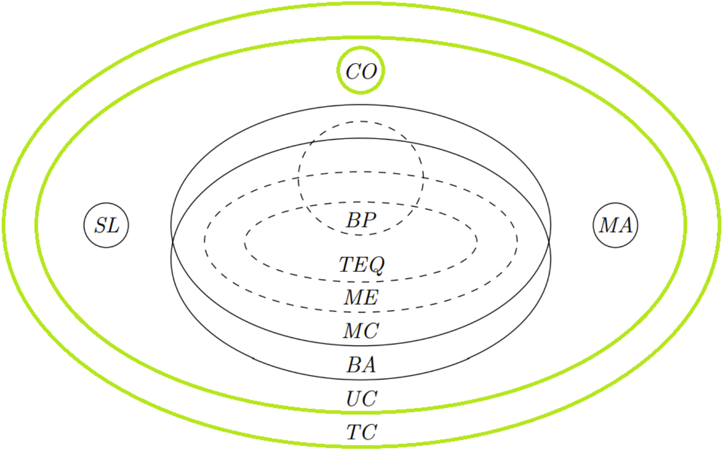
Axioms

- Condorcet-consistency: If there is a Condorcet winner x, then x is uniquely chosen.
  - *CO* ✓, *TC* ✓, *UC* ✓
- Monotonicity: If x is chosen, then it should remain chosen when it is strengthened against another alternative.
  - *CO* ✓, *TC* ✓, *UC* ✓
- Stability: A set is chosen from two different sets of alternatives if and only if it is chosen from the union of these sets.
  - *CO* ✗, *TC* ✓, *UC* ✗
- Composition-consistency: It chooses the "best" alternatives from the "best" components.
  - *CO* ✗, *TC* ✗, *UC* ✓

Computation

- The input has size $O(n^2)$, where $n =$ number of alternatives.
- Copeland set: Compute all outdegrees in time $O(n^2)$.
- Top cycle:
  - Find the strongly connected components of the tournament, and output the unique one that dominates the rest.
  - Can be done in time $O(n^2)$ by Tarjan's or Kosaraju's algorithm.
- Uncovered set:
  - Use the "can reach everything else in ≤ 2 steps" definition.
  - Multiply the adjacency matrix with itself to check reachability in 2 steps.
  - Can be done in $O(n^{2.37})$ using matrix multiplication.

Tournament Solutions



Tournament solution containment diagram [Brandt/Brill/Harrenstein '16]

Tournament Solutions

- Banks set (BA): Alternatives that appear as the maximal element of some maximal transitive subtournament.
  - Transitive tournament: The alternatives can be ordered as $a_1, \ldots, a_k$ so that a_i dominates a_j for all $i < j$.
- Slater set (SL): Alternatives that are maximal elements in some transitive tournament that can be obtained by inverting as few edges as possible.
- Bipartisan set (BP): Alternatives that are chosen with nonzero probability in the (unique) Nash equilibrium of the zero-sum game formed by the tournament matrix.
- Markov set (MC): Alternatives that stay most often in the "winner-stays" competition corresponding to the tournament.

Separation Index

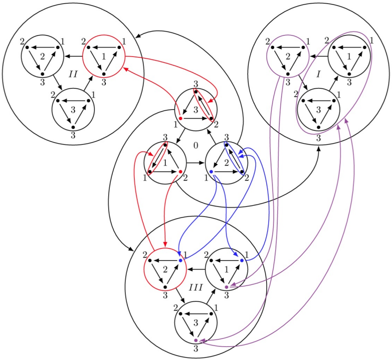
Can two given tournament solutions return disjoint sets of alternatives?
If so, what is the smallest tournament size for which this can happen?

$s \setminus d$	TC	UC	UC^∞	MC	BP	$\mathring{TC}$	BA	ME	TEQ	CO	SL	MA	KW
TC	—	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)
UC	∞	—	5 (Fig. 2)	5 (Fig. 2)	5 (Fig. 2)	5 (Fig. 2)	7[a] (Fig. 3)	5 (Fig. 2)	5 (Fig. 2)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)
UC^∞	∞	∞	—	6	6	6	5 (Fig. 2)	6	6	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)
MC	∞	∞	∞	—	6[b]	6	5 (Fig. 2)	8 (Fig. 5)	8[c] (Fig. 5)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)
BP	∞	∞	∞	∞	—	6	5 (Fig. 2)	6	6[b]	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)
$\mathring{TC}$	∞	$[11, \infty]$	$[11, \infty]$	$[11, \infty]$	$[11, \infty]$	—	5 (Fig. 2)	6	6	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)
BA	∞	∞	∞	∞	$[11, \infty]$	$[11, \infty]$	—	5 (Fig. 2)	5 (Fig. 2)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)
ME	∞	∞	∞ (Prop. 1)	∞ (Prop. 1)	$[11, \infty]$	$[11, \infty]$	∞ (Prop. 1)	—	8[d] (Fig. 5)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)
TEQ	∞	∞	∞	$[11, \infty]$	$[11, \infty]$	∞ (Prop. 1)	∞	$[11, \infty]$	—	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)
CO	∞	∞	8 (Fig. 6)	8 (Fig. 6)	8 (Fig. 6)	8 (Fig. 6)	13[e] (Fig. 3)	8 (Fig. 6)	8 (Fig. 6)	—	4 (Fig. 1)	4 (Fig. 1)	4 (Fig. 1)
SL	∞	∞	8[b]	8[b]	8[b]	8	$[11, 14]$[f]	8	8[b]	6[g]	—	5	5
MA	∞	∞	8 (Fig. 6)	8 (Fig. 6)	8 (Fig. 6)	8 (Fig. 6)	$[11, 15]$ (Fig. 3)	8 (Fig. 6)	8 (Fig. 6)	8	6	—	6
KW	∞	∞	8	8	8	8	$[11, 16]$ (Fig. 3)	8	8	7	6	7	—

Table of separation indices [Brandt/Dau/Seedig '15]

Separation Index

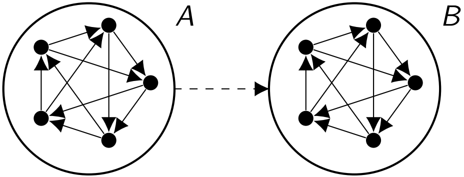
Whether *BA* and *BP* always overlap was resolved recently . . .



- Brandt/Grundbacher [2023] showed that *BA* and *BP* are disjoint for this tournament of size 36.
- The separation index of *BA* and *BP* is between 11 and 36.

Query Complexity

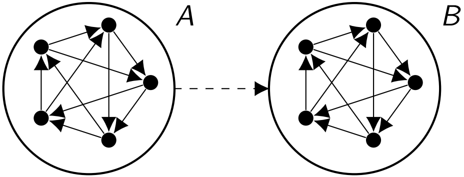
- Unlike computational complexity, the query complexity is never higher than $\Theta(n^2)$.
- However, it is $\Theta(n^2)$ for several tournament solutions [Maiti/Dey '24]
- For CO, the algorithm may need to query all edges.
  - **Idea:** Consider when all alternatives have the same outdegree.
- Proof for TC: Consider two sets A, B with $2k+1$ alternatives each.



- In each set, every alternative can reach every other alternative.

Query Complexity

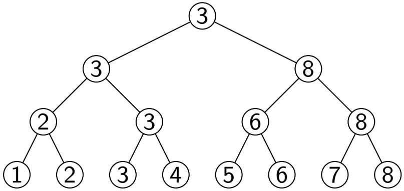
- Proof for TC: Consider two sets A, B with $2k+1$ alternatives each.



- If a query is within A or B, answer as in the figure.
- Else, answer that $a \in A$ dominates $b \in B$.
- **Claim:** All edges between A and B must be queried.
  - If all $a \in A$ dominate all $b \in B$, then $TC \subseteq A$.
  - If at least one $b \in B$ dominates at least one $a \in A$, then $TC \not\subseteq A$.
- A similar idea works for UC.
- If TC is small, all of these tournament solutions can be computed with fewer queries.

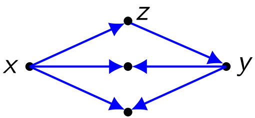
Query Complexity

- Bonus: What about deciding whether there is a Condorcet winner?
- The query complexity is exactly $2n - \lfloor \log_2 n \rfloor - 2$.
  [Balasubramanian/Raman/Srinivasaragavan '97, Procaccia '08]
- **Algorithm:**
  - Stage 1: Let alternatives compete in a balanced single-elimination tournament. Suppose the winner is x.
  - Stage 2: Let x compete against the alternatives that it has not competed against in Stage 1. Output Yes if x beats all of them.



Random Tournaments

- Several tournament solutions, including TC and UC, tend to select all alternatives in large random tournaments.
- Consider the uniform random model, where each edge is oriented in either direction with probability $1/2$, independently of other edges.
- **Proof for** UC (which implies one for TC):
  - The probability that x cannot reach y in two steps $\leq (3/4)^{n-2}$



  - By union bound over all pairs of alternatives, the probability that there exists such a pair x, y is at most $n^2 \cdot (3/4)^{n-2} \to 0$ as $n \to \infty$
  - When no such pair exists, UC is the set of all alternatives.

Condorcet Random Model

- Condorcet random model:
  - There exists an underlying linear order of players.
  - In general, a stronger player wins against a weaker player, but the weaker player upsets the stronger player with uniform probability $p \leq 1/2$.
- The uniform random model corresponds to the case $p = 1/2$.
- Łuczak/Ruciński/Gruszka [1996] showed that the top cycle selects all alternatives with high probability when $p = \omega(1/n)$, and this is tight.
- The Condorcet random model is still rather unrealistic for two important reasons.
  - In tournaments in the real world, the orientations of different edges are typically determined by different probabilities.
  - Not all probabilities of the orientation of the edges necessarily respect the ordering: "bogey teams".

Generalized Random Model

- Generalized random model:
  - The orientation of each edge is determined by probabilities within the range $[p, 1-p]$ for some parameter p, independently of other edges.
  - These probabilities are allowed to vary across edges.
- **Question:** What is the least p such that the tournament solution selects all alternatives with high probability?
- For TC we need $p \in \omega(1/n)$, while for UC we only need $p \in \Omega(\sqrt{\log n/n})$ [Saile/S. '20]
- k-kings: Alternatives that can reach every other alternative via a directed path of length $\leq k$.
- 2-kings (uncovered set) $\subseteq$ 3-kings $\subseteq \cdots \subseteq$ $(n-1)$-kings (top cycle)

Generalized Random Model

We determine how the probability threshold changes as we move from 2-kings to $(n-1)$-kings [Manurangsi/S. '22]

k-kings	Threshold p
$k = 2$	$\Omega(\sqrt{\log n/n})$
$3 \leq k \leq 4$	$\Omega(\log n/n)$
$k = 5$	$\Omega(\log\log n/n)$
$6 \leq k \leq n-2$	$\omega(1/n)$
$k = n-1$	$\omega(1/n)$

- The case $6 \leq k \leq n-2$ strengthens the previous result for $k = n-1$.
- All bounds are asymptotically tight, except for $k = 5$, where the gap is between $\Omega(\log\log n/n)$ and $\omega(1/n)$.
  - $\omega(1/n)$ is tight for the Condorcet random model.

Generalized Random Model

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$k = 5$	$\Omega(\log\log n/n)$
$6 \leq k \leq n-2$	$\omega(1/n)$
$k = n-1$	$\omega(1/n)$

- The uncovered set is clearly more selective than k-kings for $k \geq 3$.
- 3-kings and 4-kings are slightly more selective than higher-order kings.
- There is virtually no difference from $k = 5$ all the way to $k = n-1$.

Margin of Victory

- How can we differentiate between the winning alternatives?
- Brill/Schmidt-Kraepelin/S. [2020] proposed using the margin of victory (MoV).
  - Similar concepts have been applied in voting, sports modeling, political districting, etc.
- MoV(x) = minimum number of edges that need to be reversed so that x drops out of the winner set.
  - Can also define a weighted version with weighted edges.
  - The weights may represent the amount of bribe needed to change the match outcomes.
- The MoV of CO, TC, UC can be computed in polynomial time.

Margin of Victory

Theorem [Brill/Schmidt-Kraepelin/S. '20]

The MoV for CO, TC, UC can be as high as $\lfloor n/2 \rfloor$, but no higher.

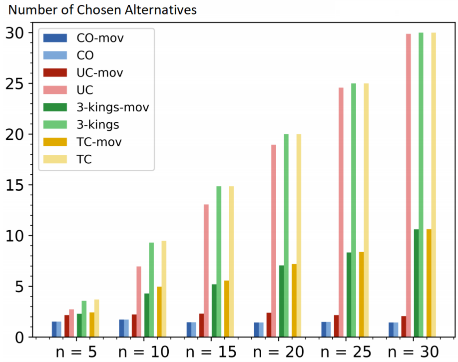
- **Upper bound:** $\mathrm{MoV}(x) \leq \lfloor n/2 \rfloor$
  - Take $y \neq x$ with the highest outdegree.
  - y has outdegree at least $\left\lfloor \frac{n-1}{2} \right\rfloor$
  - Can make y a Condorcet winner using $(n-1) - \left\lfloor \frac{n-1}{2} \right\rfloor = \lfloor n/2 \rfloor$ reversals.
- **Lower bound:** Possible that $\mathrm{MoV}(x) \geq \lfloor n/2 \rfloor$
  - Since $CO \subseteq UC \subseteq TC$, suffices to prove for CO.
  - Suppose x is a Condorcet winner (outdegree $n-1$), while the maximum outdegree of $y \neq x$ is $\left\lfloor \frac{n-1}{2} \right\rfloor$
  - Each reversal decreases $\mathrm{outdeg}(x) - \mathrm{outdeg}(y)$ by ≤ 1 (except the reversal between x and y, which decreases the difference by 2)

Margin of Victory

- Brill et al. [2021] conducted an axiomatic analysis of the MoV.
- Cover-consistency: x covers $y \Rightarrow \text{MoV}(x) \geq \text{MoV}(y)$
  - *CO* ✓, *TC* ✓, *UC* ✓
  - MoV is typically aligned with the covering relation.
- Degree-consistency: $\text{outdeg}(x) > \text{outdeg}(y) \Rightarrow \text{MoV}(x) \geq \text{MoV}(y)$
  - *CO* ✓, *TC* ✓, *UC* ✗
- Strong deg.-cons.: $\text{outdeg}(x) \geq \text{outdeg}(y) \Rightarrow \text{MoV}(x) \geq \text{MoV}(y)$
  - *CO* ✗, *TC* ✓, *UC* ✗
- MoV often provides information beyond simply the outdegrees.

Margin of Victory

- Does MoV really distinguish among winners?
- Brill et al. [2021] ran experiments to answer this question.



- On average, the number of alternatives with maximum MoV is a small fraction of the winners. ✓

Randomized Tournament Solutions

- Randomized tournament solution: Returns a probability distribution over the alternatives
- Condorcet-consistency: A Condorcet winner should receive probability 1
- k-strongly-non-manipulable-α: No group of size k can increase their combined probability by more than α

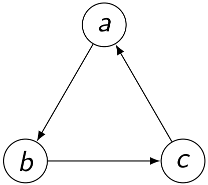
LONDON 2012: THE SUMMER OLYMPICS

Reports: Eight Badminton Players Tossed Out Of Olympics

The Disgrace of Gijon: West Germany and Austria's match of shame at 1982 World Cup

Randomized Tournament Solutions

- Observation: No Condorcet-consistent randomized tournament solution can be 2-SNM-α for any $\alpha < 1/3$.



- For any randomized tournament solution, some pair of players receive a combined probability of at most $2/3$.
- This pair of players can reverse their match outcome and increase their probability to 1.

Randomized Tournament Solutions

Theorem [Schneider/Schvartzman/Weinberg '17]

A uniformly random SE bracket is 2-SNM-1/3.

- Coupling argument: For a bracket where a pair of players could gain by manipulating, tie it with two other brackets with no manipulation potential for this pair
- Many other rules are 2-SNM-1/2 or worse!
- Randomized King-of-the-Hill:
  - If there is a Condorcet winner, declare it as the winner.
  - Else, select a player uniformly at random, and remove it along with all players that it beats. Repeat the previous step.

Theorem [Schvartzman/Weinberg/Zlatin/Zuo '20]

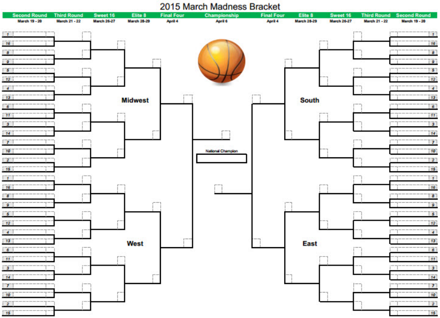
Randomized King-of-the-Hill is 2-SNM-1/3 and cover-consistent.

Randomized Tournament Solutions

- Schvartzman/Weinberg/Zlatin/Zuo (2020):
  - Assume Condorcet-consistency
  - There exists a rule that is k-SNM-2/3 for all k ✓
  - No rule can be k-SNM-1/2 for large enough k ✗
- Ding/Weinberg (2021):
  - Outcomes of matches are randomized
  - Randomized Death Match: Pick two uniformly random players, eliminate the loser, and repeat
  - This rule and Random SE Bracket perform optimally for 2-SNM
- Dinev/Weinberg (2022):
  - RDM is 3-SNM-31/60, and this is tight for this rule
- Dale/Fielding/Ramakrishnan/Sathyanarayanan/Weinberg (2022):
  - Multiple prizes according to full ranking

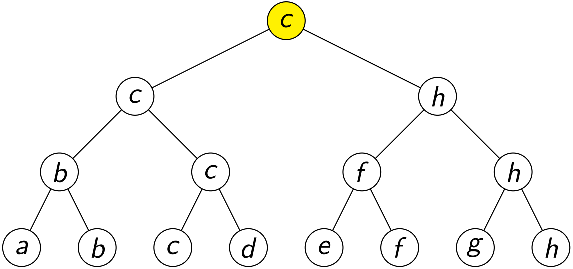
Part 2: Single-Elimination Tournaments

Single-Elimination Tournaments



- An alternative is said to be a single-elimination winner if it wins a balanced single-elimination tournament under some bracket.

Single-Elimination Tournaments



Tournament Fixing Problem

The winner of a given SE tournament can depend significantly on the initial bracket!

The Tournament Fixing Problem (TFP): Given

- A set of players
- Information for each pair of players (x, y) about whether x or y would win in a head-to-head matchup ("tournament graph")
- A player of interest v

Does there exist a bracket such that v wins the tournament?

Complexity

Theorem [Aziz/Gaspers/Mackenzie/Mattei/Stursberg/Walsh '18]

TFP is NP-complete.

- Kim/Vassilevska Williams [2015]: The problem remains NP-complete even when the player of interest v is a king that beats $n/4$ other players in the tournament graph.
- A king is a player who can reach any other player via at most 2 edges in the tournament graph.

Algorithms

Theorem [Kim/Vassilevska Williams '15]

TFP can be solved in time $O(2^n \text{poly}(n))$.

- In fact, the algorithm can also count the number of brackets under which v wins the tournament.
- **Idea:**
  - Consider all possible ways of partitioning the set of players S into two subsets T and $S \setminus T$ of equal size such that $v \in T$.
  - Iterate over all players $w \in S \setminus T$ beaten by v.
  - Compute the number of winning brackets of v in T and w in $S \setminus T$.
  - Use a fast subset convolution subroutine of Björklund/Husfeldt/Kaski/Koivisto [2007].

Algorithms

- Players can be partitioned into a constant number of types: Polynomial-time solvable [Aziz et al. '18]

- Let k be the size of a smallest feedback arc set (a set of edges whose removal leaves the tournament acyclic)
- Aziz et al. [2018]: $n^{O(k)}$ via dynamic programming
- Ramanujan/Szeider [2017]: Fixed-parameter tractable (FPT) algorithm running in time $2^{O(k^2 \log k)} n^{O(1)}$
  - Translate TFP into an algebraic system of equations and feeding it into an integer linear programming (ILP) solver
- Gupta et al. [2018]: $2^{O(k \log k)} n^{O(1)}$ via combinatorial algorithm

- Zehavi [2023]: Same running time for feedback vertex set
- **Open direction:** Other parameters, e.g., directed treewidth?

Tournament Value Maximization

- What if the organizers want to maximize the profit/popularity of the tournament?
- Tournament Value Maximization problem: Given
  - A set of players
  - Information for each pair of players (x, y) about whether x or y would win in a head-to-head matchup ("tournament graph"), and their value if they meet in a certain round

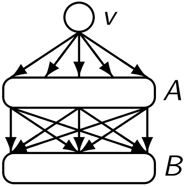
  Find a bracket that maximizes the sum of values across all matches.
- The values are round-oblivious if the value of every pair is independent of the round in which they meet.

Tournament Value Maximization

- This problem was studied by Chaudhary/Molter/Zehavi [2024]
- The problem is NP-hard (and APX-hard) when
  - All values are in $\{0, 1\}$
  - There are 3 distinct values and the values are round-oblivious
- $1/\log n$ approximation based on maximum-weight matching
- If the total value of a tournament can be determined by the number of wins of each player, there exists an $n^{O(\log n)}$ algorithm.
- If players can be classified as "popular" or "unpopular", and the value of each match equals the popularity of the winning player, there is a linear-time greedy algorithm.

Structural Results

Let v be a king.
Suppose v beats A and loses to B.



Theorem (King who beats half the players) [Vassilevska Williams '10]

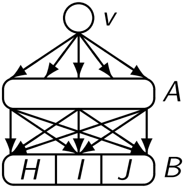
If $|A| \geq n/2$, then v is a SE winner.

Theorem (Superking) [Vassilevska Williams '10]

If every player in B loses to at least $\log_2 n$ players from A, then v is a SE winner.

Structural Results

Let v be a king.
Suppose v beats A and loses to B.



Theorem [Kim/S./Vassilevska Williams '17]

Suppose that B is a disjoint union of three sets H, I, J such that

1. $|H| < |A|$
2. Each player in I loses to at least $\log_2 n$ players in A.
3. $\text{outdeg}(j) \leq |A|$ for all $j \in J$.

Then v is a SE winner.

- Superking: $H = J = \emptyset$, King who beats $n/2$: $I = J = \emptyset$

Structural Results

Let v be a king.
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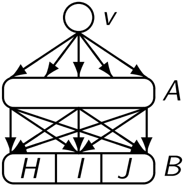

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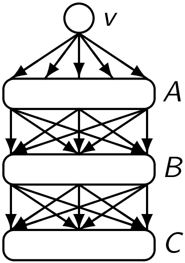
Then v is a SE winner.

- **Idea:** Match players to maintain all invariants, and apply induction

Structural Results

3-king: A player who can reach any other player via at most 3 edges in the tournament graph.



A 3-king might not be a SE winner even if it beats $n - 3$ players!
[Kim/Vassilevska Williams '15]

Structural Results

- Sufficient conditions for a 3-king to win a SE tournament.
- Kim/Vassilevska Williams [2015]:
  1. $|A| \geq n/3$
  2. Each $b \in B$ beats no more players than v does
  3. There is a perfect matching from B onto C (in particular, $|B| \geq |C|$)
- Kim/S./Vassilevska Williams [2017]:
  1. $|A| \geq n/2$
  2. Every $a \in A$ beats every $b \in B$
  3. $|B| \geq |C|$
- Any two of these three conditions are insufficient.
- **Open direction:** To what extent can we weaken these conditions?

Bribery

- Bribery-TFP (BTFP): The organizers are allowed to bribe up to b players to lose a match they would otherwise win.
  - If $b = 0$, BTFP reduces to TFP, which is NP-hard.
  - If $b = \log_2 n$, the tournament can be trivially rigged.

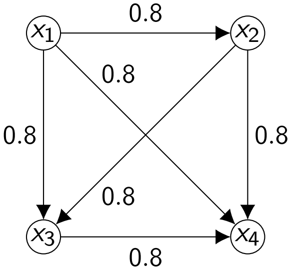
Theorem [Kim/Vassilevska Williams '15]

For any constant $\varepsilon > 0$, BTFP is NP-hard when $b \leq (1 - \varepsilon) \log_2 n$.

- Gupta/Saurabh/Sridharan/Zehavi [2019]:
  - Algorithm running in time $2^{O(k^2 \log k)} n^{O(1)}$, where k = size of a smallest feedback arc set
  - Obfuscation operations which can take in one bribery solution and output another solution in polynomial time
- Russell/Walsh [2009], Mattei/Goldsmith/Klapper/Mundhenk [2015]:
  - Bracket given in advance, but bribery is allowed

Probabilistic Approaches

- Players have varying strengths, so not all tournament graphs are equally likely to occur.
- Condorcet random model:
  - There exists an underlying linear order of players.
  - In general, a stronger player beats a weaker player, but the weaker player upsets the stronger player with probability $p \leq 1/2$.



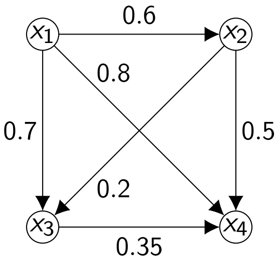
Probabilistic Approaches

- Observation: If $p \in o(\log n/n)$, the weakest player is expected to win $o(\log n)$ matches, which is insufficient to be a SE winner.
- If $p \in \Omega(\sqrt{\log n/n})$, with high probability, every player can win under some bracket [Vassilevska Williams '10]
- In fact, $p \in \Theta(\log n/n)$ is the sharp threshold! [Kim/S./Vassilevska Williams '17]

- With bribery, for any p, it suffices to bribe the top $O(\log n)$ players in the linear ordering to make any player win [Konicki/Vassilevska Williams '19]

Probabilistic Approaches

- Generalized random model: For each pair i, j, player i beats player j with probability $p_{i,j}$, independently of other pairs
  - No linear ordering of players!



Theorem [Manurangsi/S. '22]

If $p_{i,j} \in \Omega(\log n/n)$ for all i, j, then with high probability, every player can win a SE tournament under some bracket.

Probabilistic Approaches

- Probabilistic TFP (PTFP): Player i beats player j with probability $q_{i,j}$.
- The bracket must be chosen before this uncertainty is resolved.

Theorem [Chatterjee/Ibsen-Jensen/Tkadlec '16]

There exists a deterministic tournament graph such that:

- For one winning bracket of a player, the winning probability can drop by $\Theta(\varepsilon n)$ through ε-perturbations.
- For another winning bracket of this player, the drop is only $\Theta(\varepsilon \log n)$.

- The robustness can vary significantly across brackets!
- **Open question:** Suppose that the probability matrix is monotonic, i.e., $q_{i,j} \geq q_{i,j-1}$ for all $i \leq j-2$. What is the complexity of PTFP?
- If bribery is allowed, it is NP-hard [Konicki/Vassilevska Williams '19]

Seeded Setting

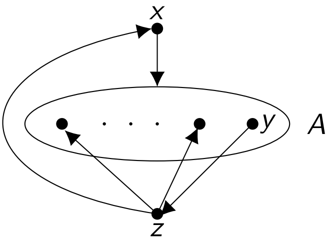
- Results on TFP so far assume that any bracket can be chosen.
- Many real-world tournaments assign seeds to players to prevent top players from meeting too early.
- For example, in ATP tennis tournaments with 32 players:
  - 8 players are assigned seeds $1, 2, \ldots, 8$.
  - The top 2 seeds cannot meet until the final.
  - The top 4 seeds cannot meet until the semifinals.
  - The top 8 seeds cannot meet until the quarterfinals.
- Let $n =$ number of players, $s =$ number of seeds

Seeded Setting

Theorem [Manurangsi/S. '23]

For any $n \geq 4$ and any s, a king who beats $n-2$ players may not be able to win a SE tournament.

Proof: Let x be our desired winner, and y and z be the top two seeds.



- Only y can beat z, so z makes the final in every bracket.
- Even if x makes the final, x will lose to z.

Seeded Setting

- Superking: A king x such that for any y who beats x, there exist at least $\log_2 n$ players who lose to x but beat y.
- Superkings can win a knockout tournament [Vassilevska Williams '10]
- This remains true if $s = 2$, but not if $s \geq 4$ [Manurangsi/S. '23]
- Ultraking: A king x such that for any y who beats x, there exist at least $n/2$ players who lose to x but beat y.

Theorem [Manurangsi/S. '23]

For any s, an ultraking can win a knockout tournament.

- This result would no longer hold if we replace $n/2$ by $n/2 - 1$ in the ultraking definition.

SE winners & Tournament Solutions

The set of SE winners can also be viewed as a tournament solution.

Theorem [Kim/S./Vassilevska Williams '17]

Every player in the Copeland set is a SE winner.

- **Proof:** Since $CO \subseteq UC$, any player in CO is a king who beats at least $n/2$ other players.

Theorem [Kim/S./Vassilevska Williams '17]

For any $0 < r < 1$, there exists a tournament such that

1. the proportion of kings who are SE winners is less than r, and
2. the proportion of SE winners who are kings is also less than r.

- Kings and SE winners are largely disjoint!

SE winners & Tournament Solutions

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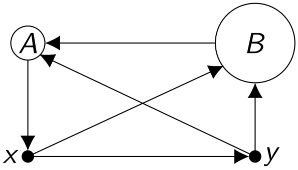

- The set of kings is $A \cup \{x, y\}$.
- If $|B| \gg |A|$ and the players in B are of roughly equal strength, then all players in B are SE winners while none of the players in A is.

SE winners & Tournament Solutions

- A SE winner may have a low Copeland score (= outdegree).

Theorem [Hulett '19]

There exists a tournament graph such that the SE winner according to a uniformly random bracket has Copeland score $n \cdot 2^{-\Theta(\sqrt{\log n})}$.

- $n \cdot 2^{-\Theta(\sqrt{\log n})}$ is lower than, say, $\Theta(n/\log n)$.
- Choosing a uniformly random alternative already yields Copeland score $(n-1)/2$.
- As an indicator of strength, the ability to win a SE tournament does not necessarily align with the Copeland score.

Future Directions

- Study other tournament formats
  - Double-elimination [Stanton/Vassilevska Williams '13, Aziz et al. '18]
  - Round-robin
  - Stepladder/challenge-the-champ [Mattei/Goldsmith/Klapper/Mundhenk '15, Yang/Dimitrov '21, Chaudhary/Molter/Zehavi '24]
  - Swiss-system [Führlich/Cseh/Lenzner '24]
  - Multi-stage tournaments
  - Promotion/relegation features
- Perform empirical studies on real-world tournaments, e.g., using data from sports competitions [Russell/van Beek '11, Mattei/Walsh '16]
- Examine the effects of the tournament structure on fairness [Ryvkin/Ortmann '08, S. '16, Arlegi/Dimitrov '20]

Let's make tournaments great again!

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